Sub-Riemannian geometry on the group of motions of the plane

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Sub-Riemannian Seminars

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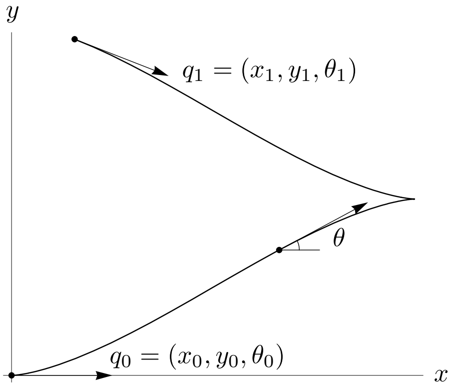
Plan of the talk

1. Problem statement
2. Geodesics
3. Local and global optimality of geodesics
4. Cut time, cut locus, spheres
5. Bicycle model, Euler elasticae, infinite geodesics
6. Isometries, homogeneous and equioptimal geodesics
7. Application to anthropomorphic curve restoration

Problem statement: Optimal motion of a mobile robot in the plane



$$q(0) = q_0, \quad q(t_1) = q_1, \quad l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \alpha^2\dot{\theta}^2}\, dt \to \min, \quad \alpha = 1$$

Group of motions (rototranslations) of a plane

$$\mathrm{SE}(2) = \mathbb{R}^2 \ltimes \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x,y) \in \mathbb{R}^2,\ \theta \in S^1 \right\}$$

Left-invariant frame on SE(2):

$$X_1(q) = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y},$$
$$X_2(q) = \frac{\partial}{\partial \theta},$$
$$X_3(q) = [X_1, X_2](q) = \sin\theta \frac{\partial}{\partial x} - \cos\theta \frac{\partial}{\partial y}.$$

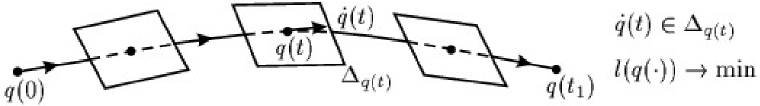
Left-invariant sub-Riemannian problem on SE(2)

$$\dot{q} = uX_1(q) + vX_2(q), \quad q \in G = \mathrm{SE}(2), \; (u, v) \in \mathbb{R}^2,$$
$$q(0) = q_0 = \mathrm{Id}, \qquad q(t_1) = q_1,$$
$$l = \int_0^{t_1} (u^2 + v^2)^{1/2}\, dt \to \min.$$

- Sub-Riemannian structure on SE(2):

$$\Delta = \mathrm{span}(X_1, X_2),$$
$$\langle X_i, X_j \rangle = \delta_{ij}, \quad i, \; j = 1, 2.$$

- Unique contact left-invariant sub-Riemannian structure on SE(2), up to local isometries and homotheties.



3D sub-Riemannian problems

- Left-invariant problem on the Heisenberg group (A.Vershik, V.Gershkovich, 1987),
- Contact problems in $\mathbb{R}^3$: local study (A.Agrachev 1996; J.-P.Gauthier 1996),
- Flat Martinet case (A.Agrachev, B.Bonnard, M.Chyba, I.Kupka 1997),
- Left-invariant problems on SO(3), SU(2), SL(2) with axisymmetric metric (V.Berestovsky, I. Zubareva 2001; U.Boscain, F.Rossi 2008),
- Left-invariant problem on SE(2) (Yu. Sachkov, I.Moiseev 2010, 2011),
- Left-invariant problem on SH(2) (Yu. Sachkov, Y.A.Butt, A.I. Bhatti 2014–2017),
- All the rest contact left-invariant 3D sub-Riemannian structures (one-parameter family) not studied, integrable in elliptic integrals of the 3-rd kind.

Existence of solutions and Pontryagin maximum principle

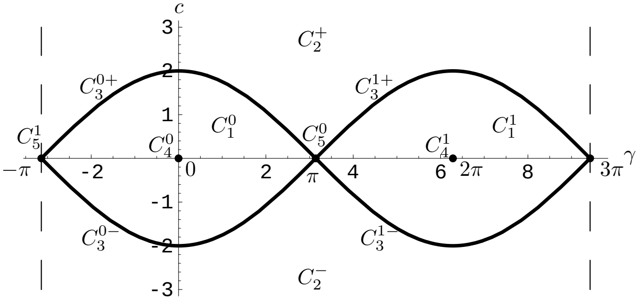
- Complete controllability by Rashevskii-Chow theorem
- Existence of minimizers by Filippov's theorem
- Abnormal extremal trajectories constant.
- Normal extremals:

$$\dot{\gamma} = c, \quad \dot{c} = -\sin\gamma, \qquad (\gamma, c) \in C = T^*_{\mathrm{Id}} G \cap \{H = 1/2\} \cong (2S^1_\gamma) \times \mathbb{R}_c,$$
$$\dot{x} = \sin\frac{\gamma}{2}\cos\theta, \quad \dot{y} = \sin\frac{\gamma}{2}\sin\theta, \quad \dot{\theta} = -\cos\frac{\gamma}{2}.$$

- Integrable in Jacobi's elliptic functions.

Stratification of phase cylinder of pendulum $C = \cup_{i=1}^{5} C_i$

- Energy integral $E = c^2/2 - \cos\gamma \in [-1, +\infty)$
- $C_1 = \{\lambda \in C \mid E \in (-1, 1)\}$,
- $C_2 = \{\lambda \in C \mid E \in (1, +\infty)\}$,
- $C_3 = \{\lambda \in C \mid E = 1,\ c \neq 0\}$,
- $C_4 = \{\lambda \in C \mid E = -1\}$,
- $C_5 = \{\lambda \in C \mid E = 1,\ c = 0\}$.



Parametrisation of geodesics

- $\lambda = (\gamma, c) \in C_1 \quad \Rightarrow$

$$\begin{aligned} \theta_t &= s_1(\operatorname{am}\varphi - \operatorname{am}\varphi_t) \pmod{2\pi}, \\ x_t &= (s_1/k)[\operatorname{cn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}\varphi_t) + \operatorname{sn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi_t))], \\ y_t &= (1/k)[\operatorname{sn}\varphi(\operatorname{dn}\varphi - \operatorname{dn}\varphi_t) - \operatorname{cn}\varphi(t + \operatorname{E}(\varphi) - \operatorname{E}(\varphi_t))]. \end{aligned}$$

- $\lambda = (\gamma, c) \in C_2 \quad \Rightarrow$

$$\begin{aligned} \cos\theta_t &= k^2 \operatorname{sn}\psi \operatorname{sn}\psi_t + \operatorname{dn}\psi \operatorname{dn}\psi_t, \\ \sin\theta_t &= k(\operatorname{sn}\psi \operatorname{dn}\psi_t - \operatorname{dn}\psi \operatorname{sn}\psi_t), \\ x_t &= s_2 k[\operatorname{dn}\psi(\operatorname{cn}\psi - \operatorname{cn}\psi_t) + \operatorname{sn}\psi(t/k + \operatorname{E}(\psi) - \operatorname{E}(\psi_t))], \\ y_t &= s_2[k^2 \operatorname{sn}\psi(\operatorname{cn}\psi - \operatorname{cn}\psi_t) - \operatorname{dn}\psi(t/k + \operatorname{E}(\psi) - \operatorname{E}(\psi_t))]. \end{aligned}$$

- $\lambda = (\gamma, c) \in C_3 \cup C_4 \cup C_5 \quad \Rightarrow \quad$ hyperbolic and linear functions.

Geodesics: generic cases

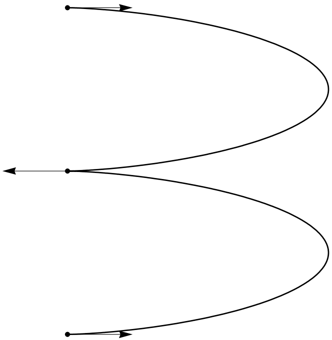


Figure: Non-inflectional curves, $\lambda \in C_1$

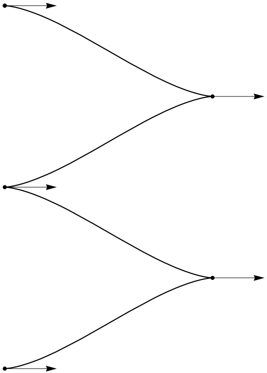


Figure: Inflectional curves, $\lambda \in C_2$

Geodesics: special cases

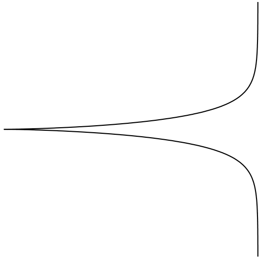


Figure: Tractrix, $\lambda \in C_3$

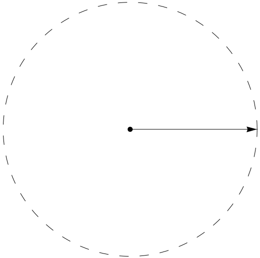


Figure: e^{tX_2}, $\lambda \in C_4$

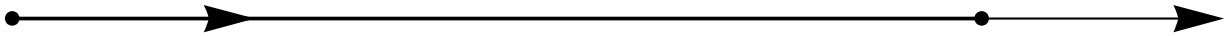


Figure: e^{tX_1}, $\lambda \in C_5$

Optimality of geodesics

$q(t)$ is locally optimal:

$$\exists \varepsilon > 0 \quad \forall \text{ arclength-parameterized trajectory } \widetilde{q}:$$
$$\|\widetilde{q} - q\|_C < \varepsilon,\ q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1$$

$q(t)$ is globally optimal:

$$\forall \text{ arclength-parameterized trajectory } \widetilde{q}:$$
$$q(0) = \widetilde{q}(0),\ q(t_1) = \widetilde{q}(\tilde{t}_1) \quad \Rightarrow \quad t_1 \leq \tilde{t}_1$$

Loss of optimality

- Short arcs of extremal trajectories are optimal (extremal trajectories are geodesics).
- Cut time: $t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s) \text{ is optimal for } s \in [0, t]\}$.
- Reasons for loss of optimality:
  (1) Maxwell point q_t: $\exists$ geodesic $\widetilde{q}_s \neq q_s$: $q_0 = \widetilde{q}_0, \ q_t = \widetilde{q}_t$
  (2) Conjugate point: $q_t \in$ envelope of the family of geodesics

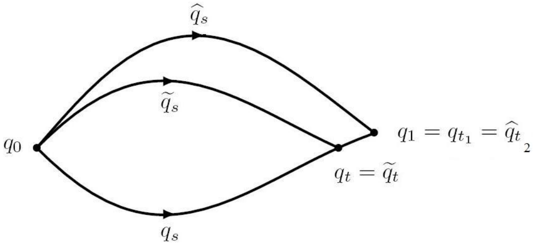


Figure: Maxwell point

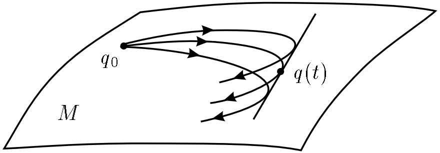


Figure: Conjugate point

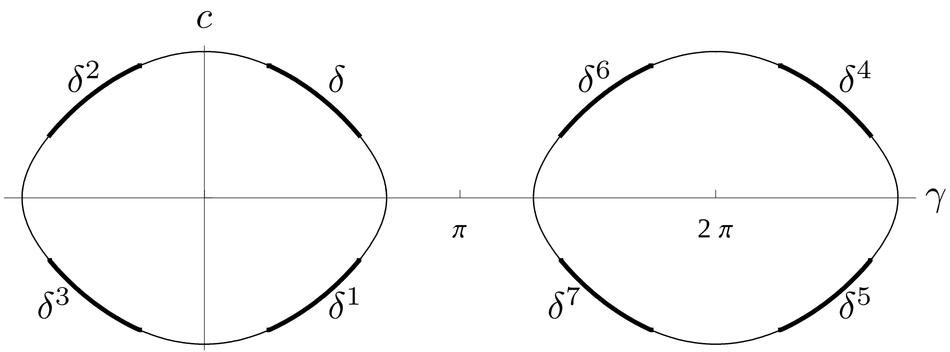
- $t_{\text{cut}} = \min(t_{\text{Max}}, t_{\text{conj}})$

Reflections ε^i in the phase cylinder of pendulum $\ddot{\gamma} = -\sin\gamma$

- Group of symmetries of parallelepiped

$$\mathrm{Sym} = \{\mathrm{Id}, \varepsilon^1, \dots, \varepsilon^7\} = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2.$$

- Action of reflections $\varepsilon^i \ : \ \delta \mapsto \delta^i$ on trajectories of pendulum:



Action of reflections ε^i on curves (x_t, y_t) modulo rotations

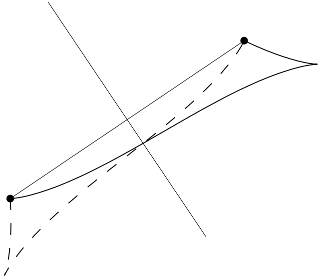


Figure: ε^1, ε^2

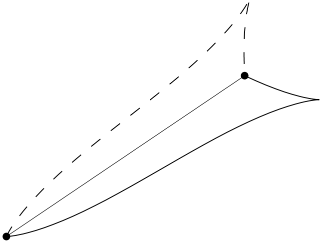


Figure: ε^4, ε^7

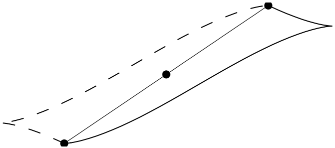
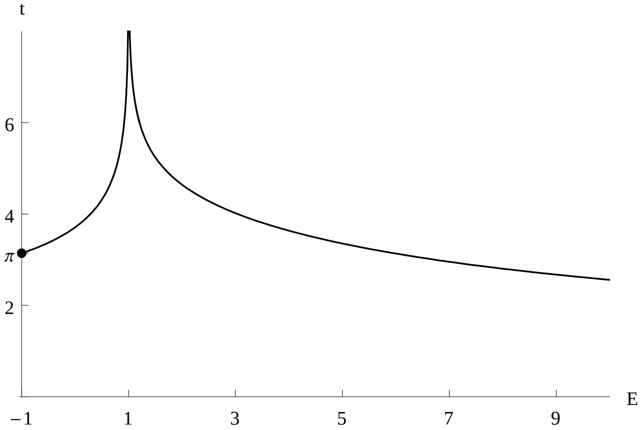


Figure: ε^5, ε^6

Maxwell points corresponding to reflections

- Fixed points of reflections ε^i: $\qquad t = t^n_{\varepsilon^i}, \qquad i = 1, 2, \dots, 7, \quad n = 1, 2, \dots$
- Upper bound of cut time: $t_{\text{cut}} \leq t^{\mathrm{Sym}}_{\text{Max}} := \min(t^1_{\varepsilon^i})$.
- Plot of function $\mathbf{t} = t^{\mathrm{Sym}}_{\text{Max}}(E)$:



Exponential mapping and conjugate points

- Exponential mapping

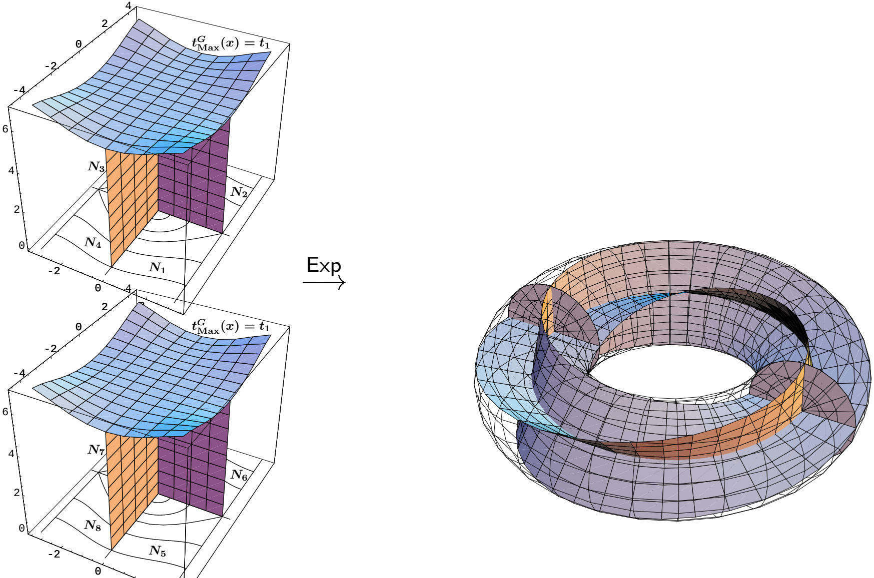
$$\begin{aligned} \mathrm{Exp} &: (\lambda, t) = (\gamma, c, t) \mapsto q(t), \\ \mathrm{Exp} &: N = C \times \mathbb{R}_+ \to M, \qquad C = (T^*_{\mathrm{Id}} G) \cap \{H = 1/2\} \end{aligned}$$

- q — conjugate point $\iff$ q — critical value of Exp
- The first conjugate time

$$t^1_{\mathrm{conj}}(\lambda) = \min\{t > 0 \mid q(t) = \mathrm{Exp}(\lambda, t) \text{ conjugate point }\}.$$

- $\lambda \in C_1 \cup C_3 \cup C_4 \cup C_5 \quad \Rightarrow \quad t^1_{\mathrm{conj}}(\lambda) = +\infty.$
- $\lambda \in C_2 \quad \Rightarrow \quad t^1_{\varepsilon^6}(\lambda) \geq t^1_{\mathrm{conj}}(\lambda) \geq t^1_{\varepsilon^2}(\lambda) = t^{\mathrm{Sym}}_{\mathrm{Max}}(\lambda).$
- $\Rightarrow$ Global bound:

$$t^1_{\mathrm{conj}}(\lambda) \geq t^{\mathrm{Sym}}_{\mathrm{Max}}(\lambda), \qquad \lambda \in C.$$



$t^G_{\mathrm{Max}}(x) = t_1$
N_3
N_2
N_4
N_1
$t^G_{\mathrm{Max}}(x) = t_1$
N_7
N_6
N_8
N_5
Exp

Cut time and cut points

$$t_{\text{cut}}(\lambda) = t_{\text{Max}}^{\text{Sym}}(\lambda) = \begin{cases} t^1_{\varepsilon^5} = 2K(k) = T/2, & \lambda \in C_1, \\ t^1_{\varepsilon^2} = 2kp^1_1(k) \in (T, 2T), & \lambda \in C_2, \\ +\infty, & \lambda \in C_3 \cup C_5, \\ t^1_{\varepsilon^5} = \pi = T/2, & \lambda \in C_4 \end{cases}$$

$$p = p^1_1(k) \quad : \quad \text{cn}(p,k)(\text{E}(p,k) - p) - \text{dn}(p,k)\,\text{sn}(p,k) = 0$$

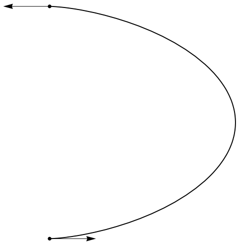


Figure: $\lambda \in C_1$

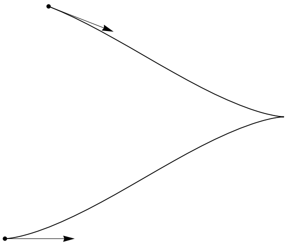


Figure: $\lambda \in C_2$

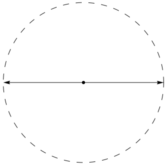


Figure: $\lambda \in C_4$

Infinite geodesics

Definition

A geodesic $q(t)$, $t \in \mathbb{R}$, is called *an infinite geodesic* if any its subarc $q(t)$, $t \in [a, b]$, is a minimizer.

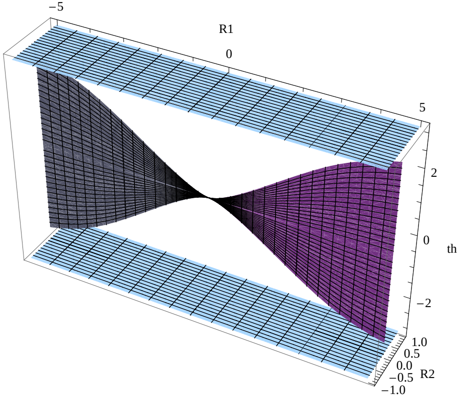
Proposition

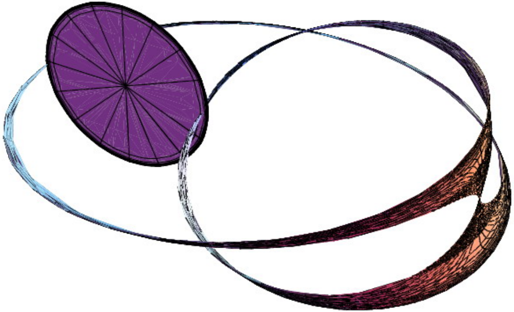
A geodesic in $\mathrm{SE}(2)$ *is an infinite geodesic if and only if its projection to the plane* (x, y) *is:*

(1) *a line, or*

(2) *a tractrix.*

Cut locus

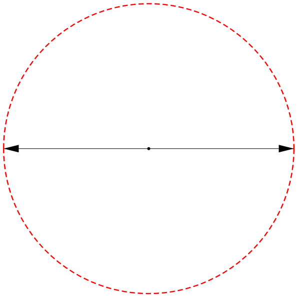




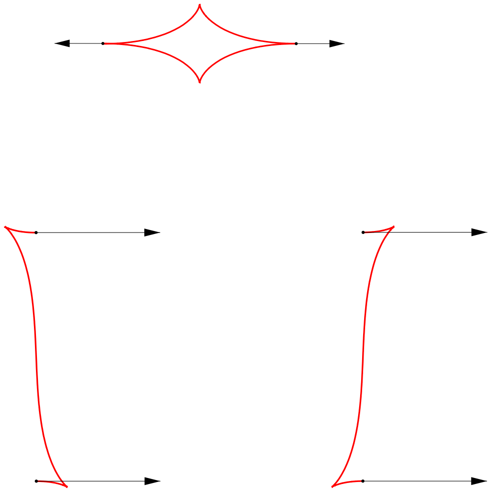
Minimizers — one-parameter subgroups

$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = 0 \qquad \Rightarrow \qquad q(t) = e^{tX_1}:$$

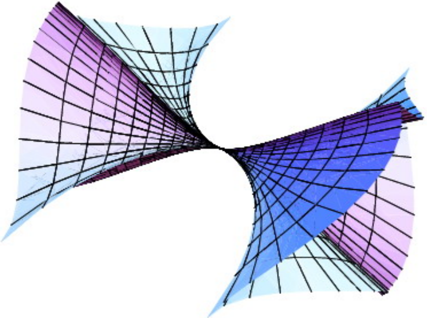
$$x_1 = 0, \qquad y_1 = 0, \qquad \theta_1 \neq 0 \qquad \Rightarrow \qquad q(t) = e^{tX_2}:$$

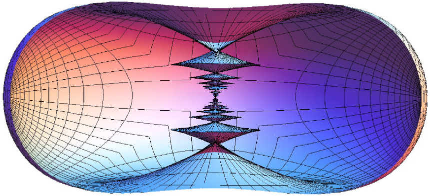


Minimizers for special boundary conditions



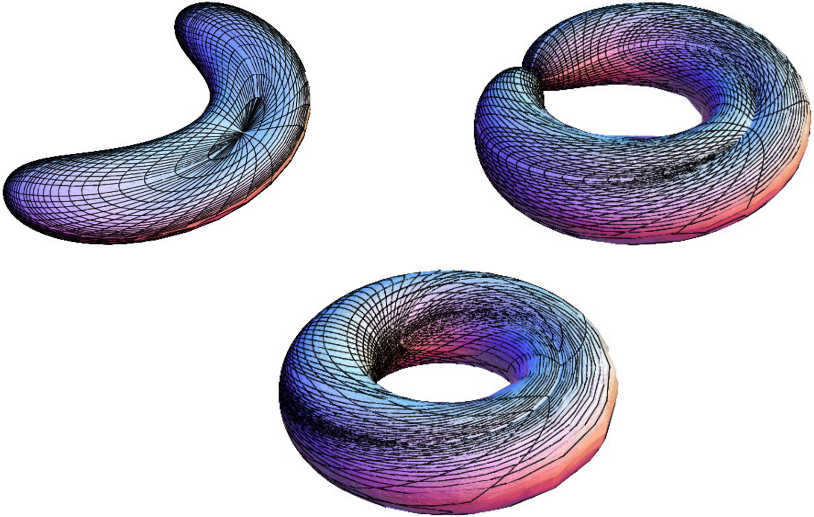
Caustic and wavefront for $R = \pi$



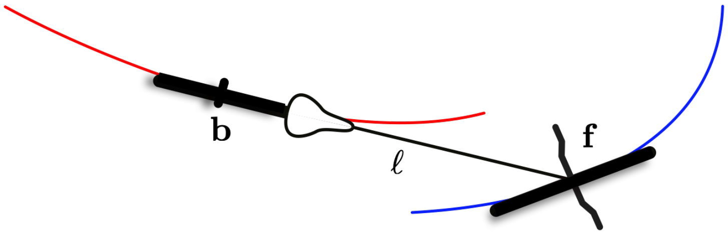


Global structure of sub-Riemannian spheres:

$$R < \pi, \quad R = \pi, \quad R > \pi$$



Model of bicycle

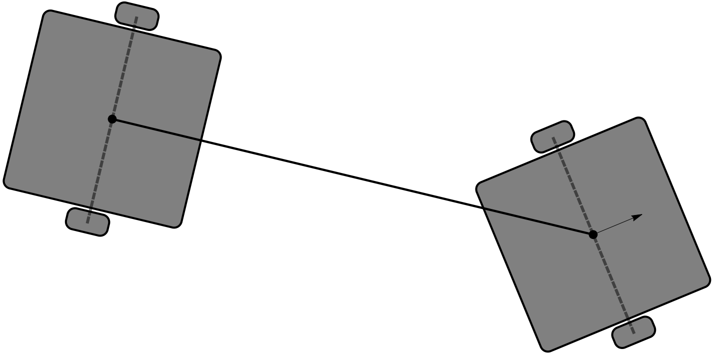


- Configuration space:

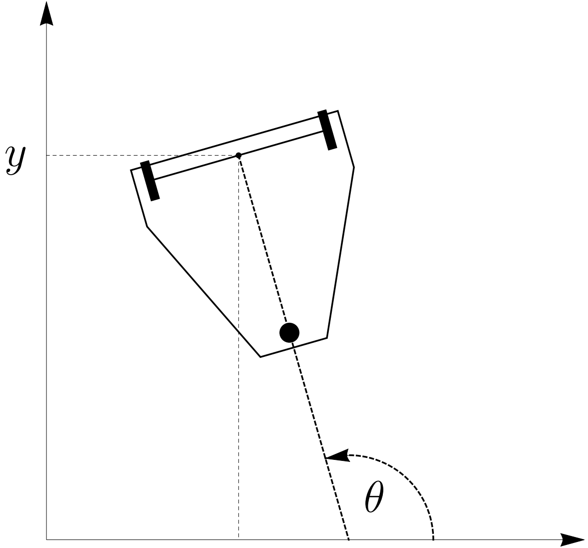
$$Q = \{(\mathbf{b}, \mathbf{f}) \in \mathbb{R}^2 \times \mathbb{R}^2 \mid \|\mathbf{f} - \mathbf{b}\| = 1\} \cong \mathbb{R}^2 \times S^1 \cong \mathrm{SE}(2),$$
$$\ell = 1.$$

- No-skid condition: $\forall t \; \exists a \in \mathbb{R} : \; \dot{\mathbf{b}} = a(\mathbf{f} - \mathbf{b})$.
- What are the shortest paths of the front wheel $\mathbf{f}$ of the bicycle?

Car-like robot with trailer



Car-like robot with two driving wheels and a spherical wheel



Rickshaw

Sub-Riemannian problem $(\mathbf{f})$ for the front wheel

State space:

$$\mathrm{SE}(2) = \{q_{\mathbf{f}} = (x_{\mathbf{f}}, y_{\mathbf{f}}, \theta_{\mathbf{b}}) \in \mathbb{R}^2_{\mathbf{f}} \times S^1_{\mathbf{b}}\}.$$

Control system:

$$\begin{aligned} \dot{x}_{\mathbf{f}} &= u_1, \\ \dot{y}_{\mathbf{f}} &= u_2, \\ \dot{\theta}_{\mathbf{b}} &= -u_1 \sin\theta_{\mathbf{b}} + u_2 \cos\theta_{\mathbf{b}}, \end{aligned}$$

where $(u_1, u_2) \in \mathbb{R}^2$.
Boundary conditions:

$$q_{\mathbf{f}}(0) = q^0_{\mathbf{f}}, \qquad q_{\mathbf{f}}(t_1) = q^1_{\mathbf{f}}.$$

Sub-Riemannian length:

$$l_{\mathbf{f}} = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \to \min.$$

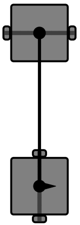
Sub-Riemannian problem (b) for the back wheel

- State space:
$$\mathrm{SE}(2) = \{q_{\mathbf{b}} = (x_{\mathbf{b}}, y_{\mathbf{b}}, \theta_{\mathbf{b}}) \in \mathbb{R}^2_{\mathbf{b}} \times S^1_{\mathbf{b}}\}.$$
- Control system:
$$\begin{aligned} \dot{x}_{\mathbf{b}} &= u_{\mathbf{b}} \cos\theta_{\mathbf{b}}, \\ \dot{y}_{\mathbf{b}} &= u_{\mathbf{b}} \sin\theta_{\mathbf{b}}, \\ \dot{\theta}_{\mathbf{b}} &= v_{\mathbf{b}}, \end{aligned}$$
where $(u_{\mathbf{b}}, v_{\mathbf{b}}) \in \mathbb{R}^2$.
- Boundary conditions:
$$q_{\mathbf{b}}(0) = q^0_{\mathbf{b}}, \qquad q_{\mathbf{b}}(t_1) = q^1_{\mathbf{b}}.$$
- Sub-Riemannian length:
$$l_{\mathbf{b}} = \int_0^{t_1} \sqrt{u^2_{\mathbf{b}} + v^2_{\mathbf{b}}}\, dt \to \min.$$
- Exactly the sub-Riemannian problem on SE(2) studied above.

Isomorphism between the problems (**b**) and (**f**)

$$
\begin{aligned}
x_{\mathbf{b}} &= x_{\mathbf{f}} - \cos\theta_{\mathbf{b}}\,,\\
y_{\mathbf{b}} &= y_{\mathbf{f}} - \sin\theta_{\mathbf{b}}\,,\\
u_{\mathbf{b}} &= u_1 \cos\theta_{\mathbf{b}} + u_2 \sin\theta_{\mathbf{b}}\,,\\
v_{\mathbf{b}} &= -u_1 \sin\theta_{\mathbf{b}} + u_2 \cos\theta_{\mathbf{b}}\,,\\
l_{\mathbf{b}} &= l_{\mathbf{f}}\,.
\end{aligned}
$$

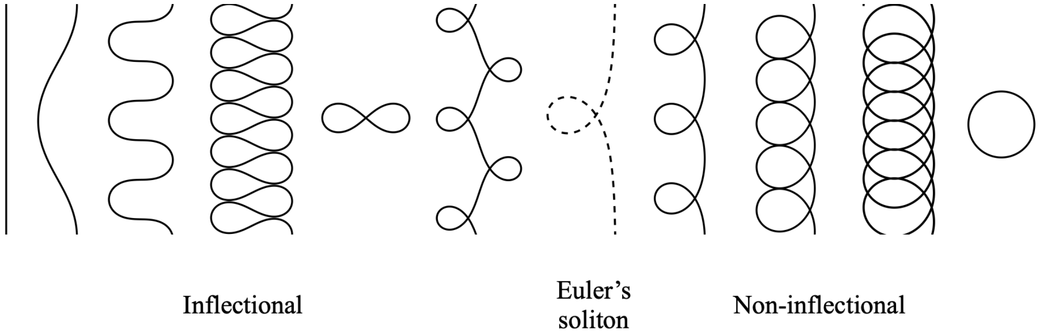
Example of correspondence between solutions to the problems (**f**) and (**b**): line and tractrix



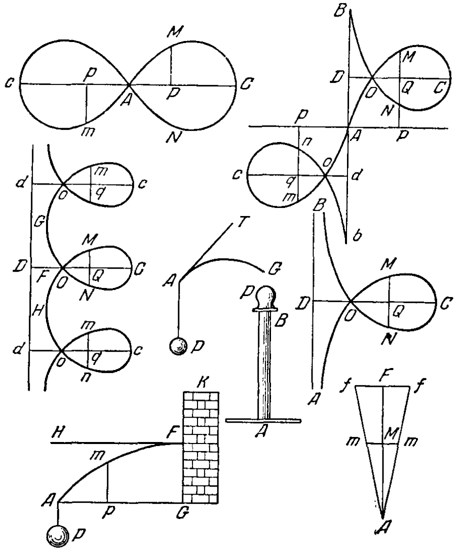
Geodesics for the problem $(\mathbf{f})$

Theorem

Projection $(x_{\mathbf{f}}, y_{\mathbf{f}})$ of a geodesic for the problem $(\mathbf{f})$ is either a straight line, or a non-inflectional Euler's elastica *of width ≤ 2, including circle and Euler's soliton. Any form of a non-inflectional elastica is possible.*



Euler's sketches of elasticae (1744)



Definitions of Euler's elasticae

1. Elastica is a planar curve whose curvature is a linear function of its distance to a fixed line in the plane.
2. Elastica is a planar curve whose curvature κ satisfies the ODE

$$\ddot{\kappa} + \frac{1}{2}\kappa^3 + A\kappa = 0, \qquad A \equiv \text{const}\,.$$

3. Elastica is a curve (x, y) that satisfies the ODEs

$$\begin{aligned} \dot{x} &= \cos\theta, \\ \dot{y} &= \sin\theta, \\ \ddot{\theta} &= -r\sin\theta, \qquad r \equiv \text{const}\,. \end{aligned}$$

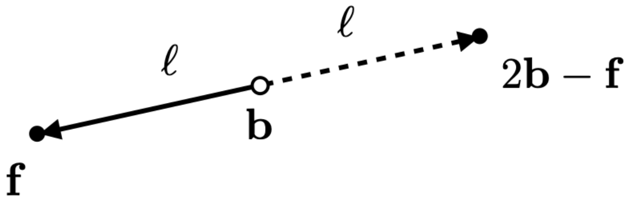
Flipping a bike about its back wheel

Lemma

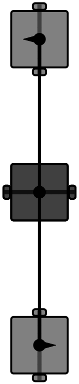
The map

$$\Phi \;:\; Q \to Q, \qquad (\mathbf{b}, \mathbf{f}) \mapsto (\mathbf{b}, 2\mathbf{b} - \mathbf{f})$$

which 'flips' the bike frame about the back wheel is a global sub-Riemannian isometry of Q.



Flipping the line to Euler's soliton

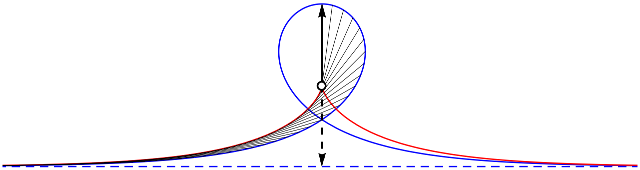


Infinite geodesics in the problem **(f)**

An infinitely long bike path is an infinite geodesic if and only if it is one of the following two types:

(1) its front track is a straight line and its back track is a tractrix or a straight line, or
(2) its front track is an Euler soliton of width twice the bike length and its back track is a tractrix.

There is an isometric involution of the bicycle configuration space which takes paths of one type to paths of the other, provided the back track of the path is a tractrix and not a line.



The two infinite minimizing bike paths share the tractrix (red) as a common back track; the two front tracks are a straight line (dashed blue) and an 'Euler's soliton' (solid blue).

Isometries

Theorem

The group $\mathrm{Isom}(Q)$ *of all isometries of* Q *is an extension of* $\mathrm{E}(2)$ *by the two-element group* $\mathbb{Z}_2$. *This two-element group is generated by the isometric involution* Φ *which 'flips the bike frame'. Thus*

$$\mathrm{Isom}(Q) \simeq \mathrm{E}(2) \rtimes \mathbb{Z}_2 \simeq \mathrm{SE}(2) \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2).$$

The identity component of $\mathrm{Isom}(Q)$ *is* $\mathrm{SE}(2)$. $\mathrm{SE}(2)$ *acts freely and transitively on* Q *and so induces a sub-Riemannian isometry between* Q *and a left-invariant sub-Riemannian metric on* $\mathrm{SE}(2)$.

Definition

A geodesic γ in a sub-Riemannian manifold M is called *homogeneous*, if it is a homogeneous space of a one-parameter subgroup $\{\varphi_s \mid s \in \mathbb{R}\} \subset \mathrm{Isom}(M)$:

1. $\forall\, s \in \mathbb{R} \quad \varphi_s(\gamma) \subset \gamma$,
2. $\forall\, g_1, g_2 \in \gamma \quad \exists\, s \in \mathbb{R} \ : \quad \varphi_s(g_1) = g_2$.

A sub-Riemannian manifold is called *geodesically orbital* if all its geodesics are homogeneous.

Equioptimal geodesics

Definition

Let $\{q(t) \mid t \geq 0\} \subset M$ be a geodesic. An arc $\{q(t) \mid t \in [0, T]\}$ is called a *non-extendable minimizer* if it is a minimiser, but any arc $\{q(t) \mid t \in [0, T + \varepsilon]\}$, $\varepsilon > 0$, is not a minimizer.

In other words, $T = t_{\mathrm{cut}}(q(\cdot))$.

Definition

An arclength-parameterized geodesic $q(t)$, $t \in \mathbb{R}$, is called *equioptimal*, if it satisfies the following property:

if $q(t)$, $t \in [0, T]$, is a non-extendable minimiser, then for any $\tau \in \mathbb{R}$ the geodesic $q(t + \tau)$, $t \in [0, T]$, is also a non-extendable minimiser.

A sub-Riemannian manifold M is called *equioptimal*, if any its arclength-parameterized geodesic is equioptimal.

Examples of equioptimal Lie groups

Left-invariant sub-Riemannian structures on:

- The Heisenberg group
- SO(3), SU(2), SL(2) with axisymmetric metric,
- SE(2), SH(2),
- The Engel and Cartan groups.

In other words, all left-invariant sub-Riemannian structures where the cut time is known.

Why?

Proposition

If a sub-Riemannian geodesic is homogeneous, then it is equioptimal.

Homogeneous geodesics in SE(2)

Theorem

The only homogeneous geodesics in SE(2) *are one-parameter subgroups that are geodesics:*

$$e^{tX_1} \qquad \text{and} \qquad e^{tX_2}.$$

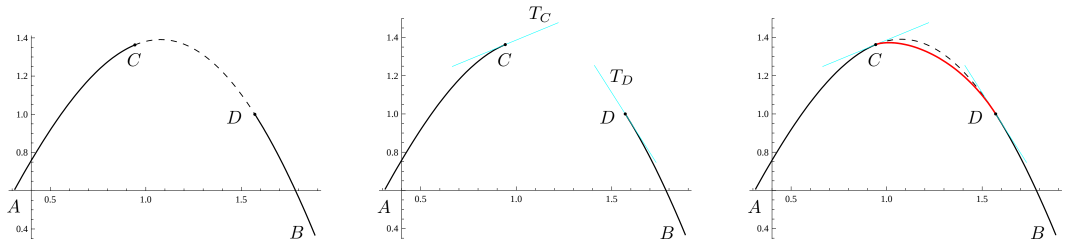
Thus SE(2) *is not geodesically orbital, although equioptimal.*

The following Lie groups are geodesically orbital and equioptimal:

- The Heisenberg group
- SO(3), SU(2), SL(2) with axisymmetric metric.

What is the geometric reason of equioptimality in the non-geodesically orbital case?

Application: Anthropomorphic restoration of isophotes and neurogeometry



[1] J.Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, *J. Physiology - Paris* 97 (2003), 265–309

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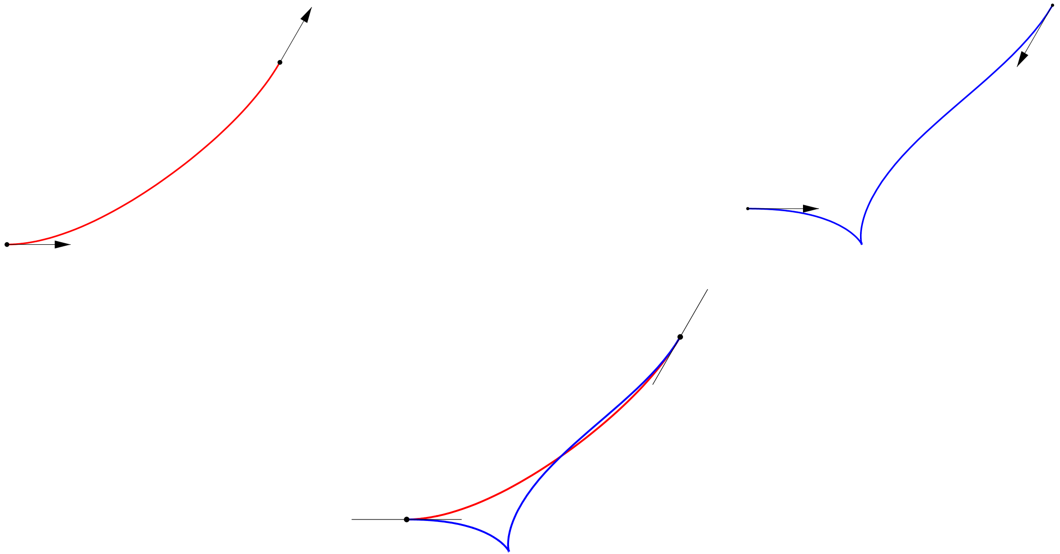
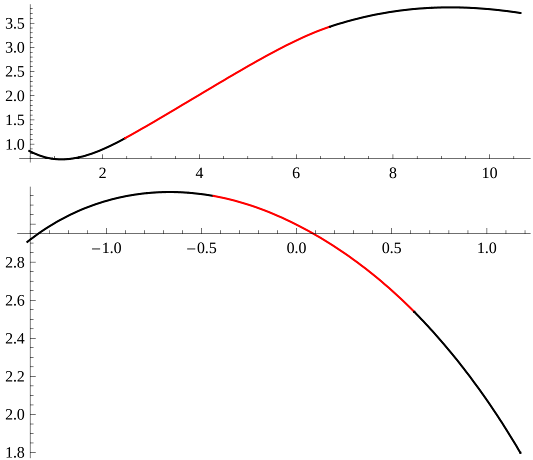
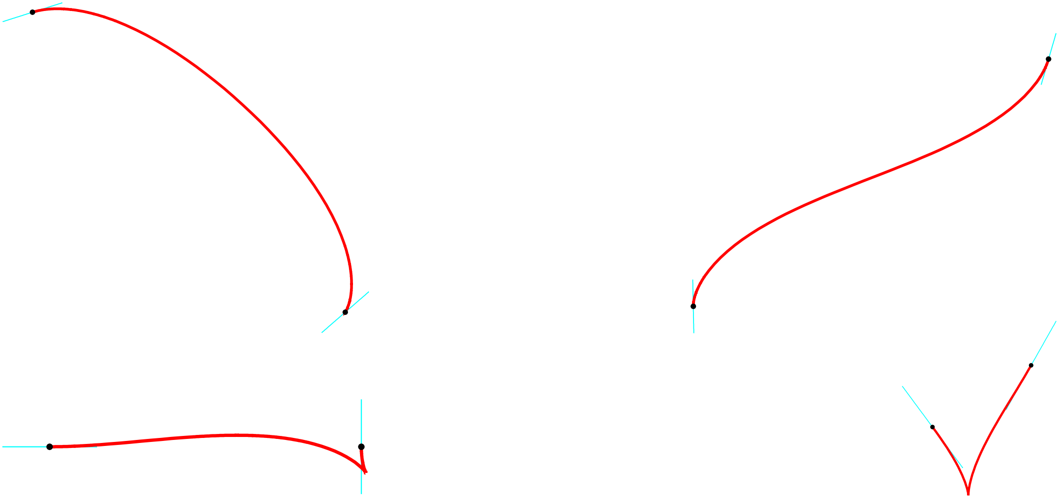
Optimal solution for the problem on $\mathbb{R}^2 \times \mathbb{R}P^1$



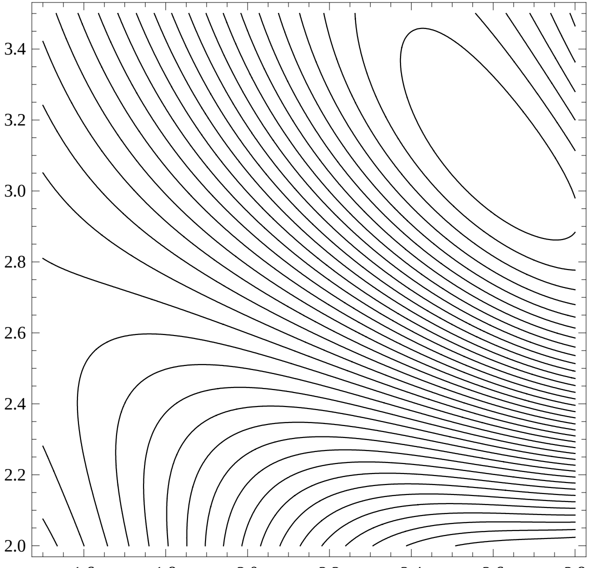
Restored curves



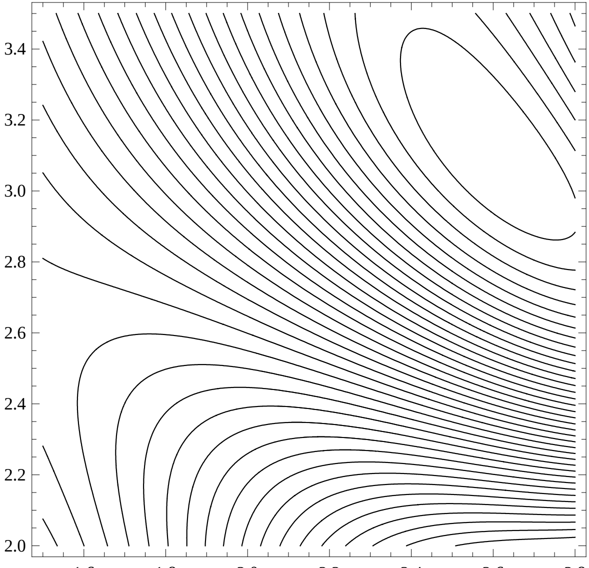
Smooth and non-smooth arcs



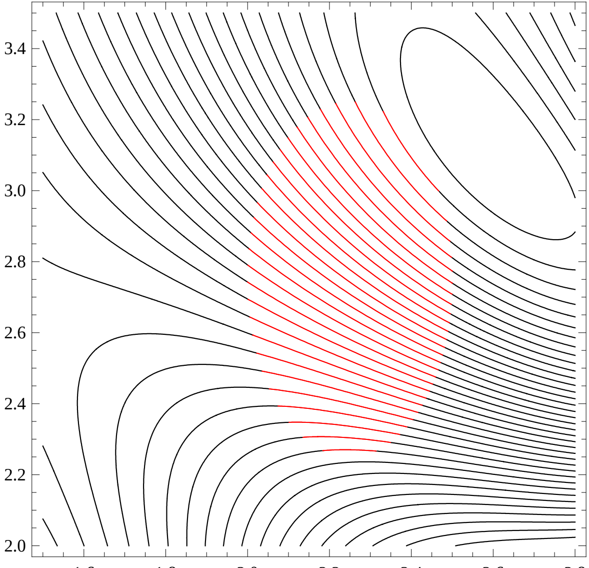
Initial family of isophotes



Restored family of isophotes



Restored family of isophotes



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[1] I. Moiseev, Yu. L. Sachkov, Maxwell strata in sub-Riemannian problem on the group of motions of a plane, *ESAIM: COCV*, 16 (2010), 380–399

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[4] A.Ardentov, A.Mashtakov, Yu. Sachkov, Parallel Algorithm and Software for Image Inpainting via Sub-Riemannian Minimizers on the Group of Rototranslations, *Numer. Math. Theor. Meth. Appl.*, 2013, Vol. 6, No. 1, pp. 95-115.

[5] A. Ardentov, G. Bor, E. Le Donne, R. Montgomery, Yu. Sachkov, Bicycle paths, elasticae and sub-Riemannian geometry, *https://arxiv.org/abs/2010.04201*

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