# A Cortical Based Model for Contour Completion on the Retinal Sphere

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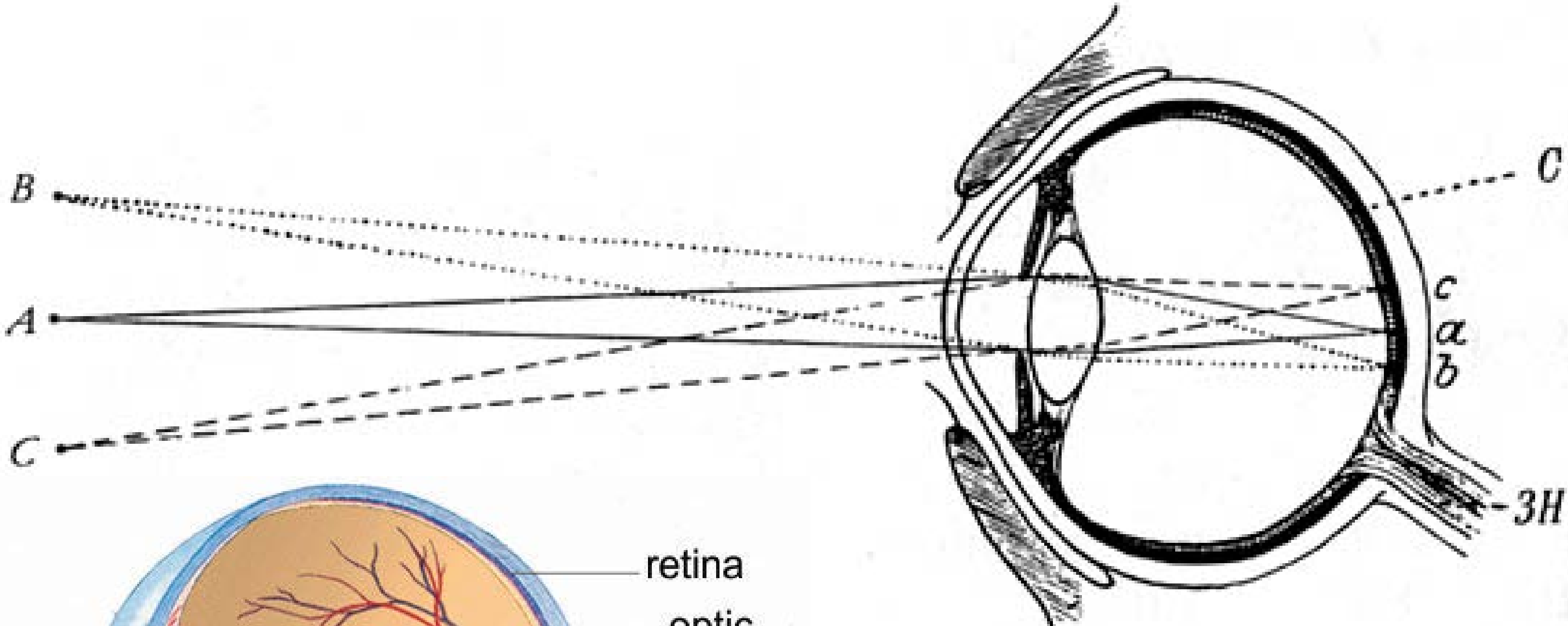
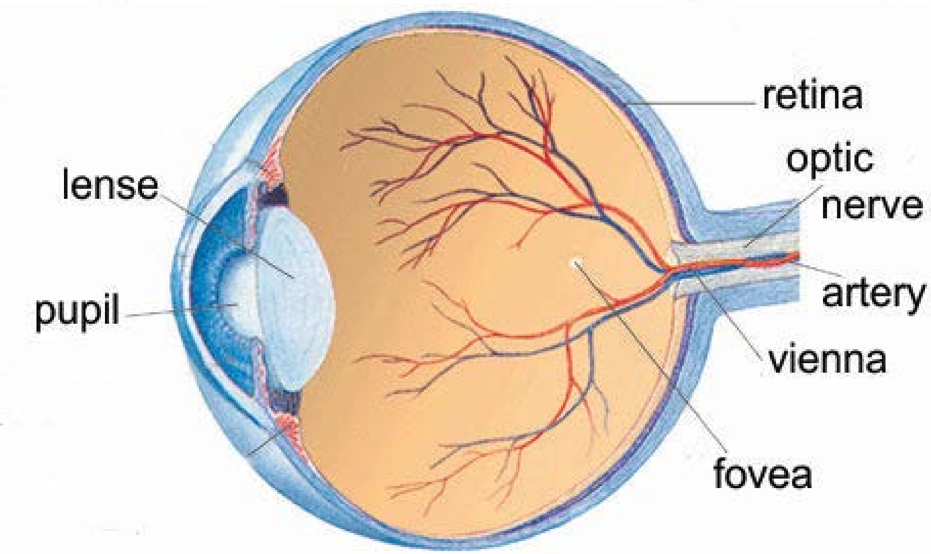
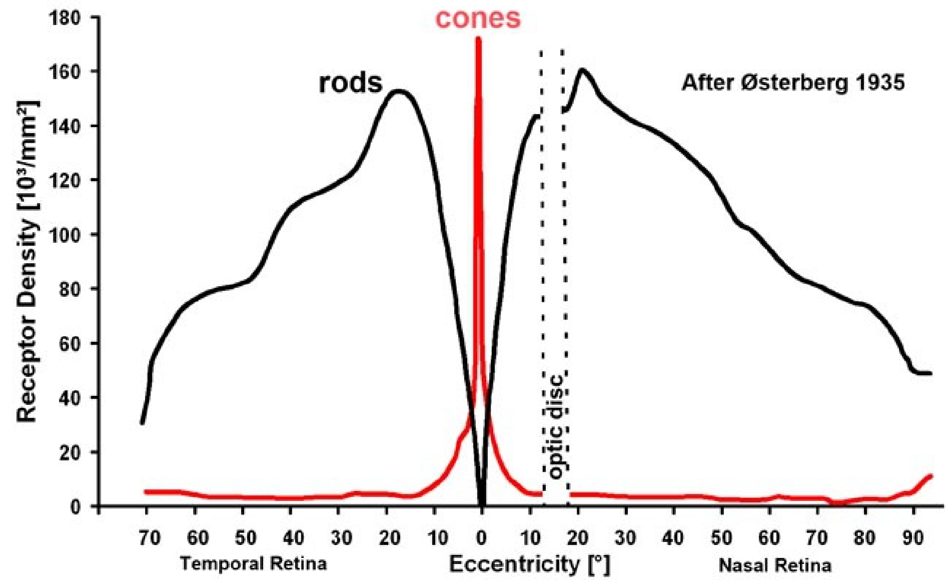
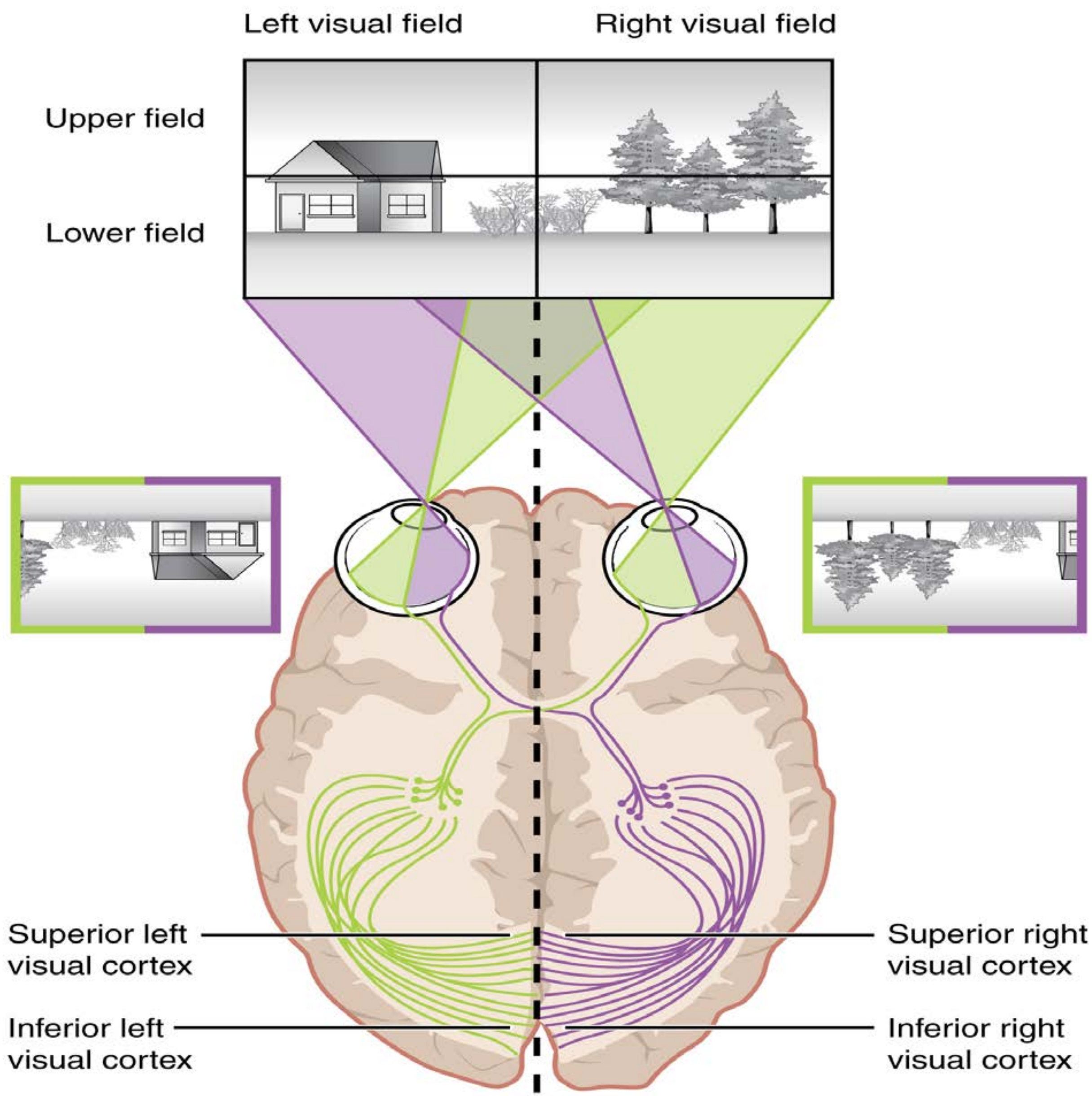
# Human Eye


B
A
C
C
c
a
b
3H

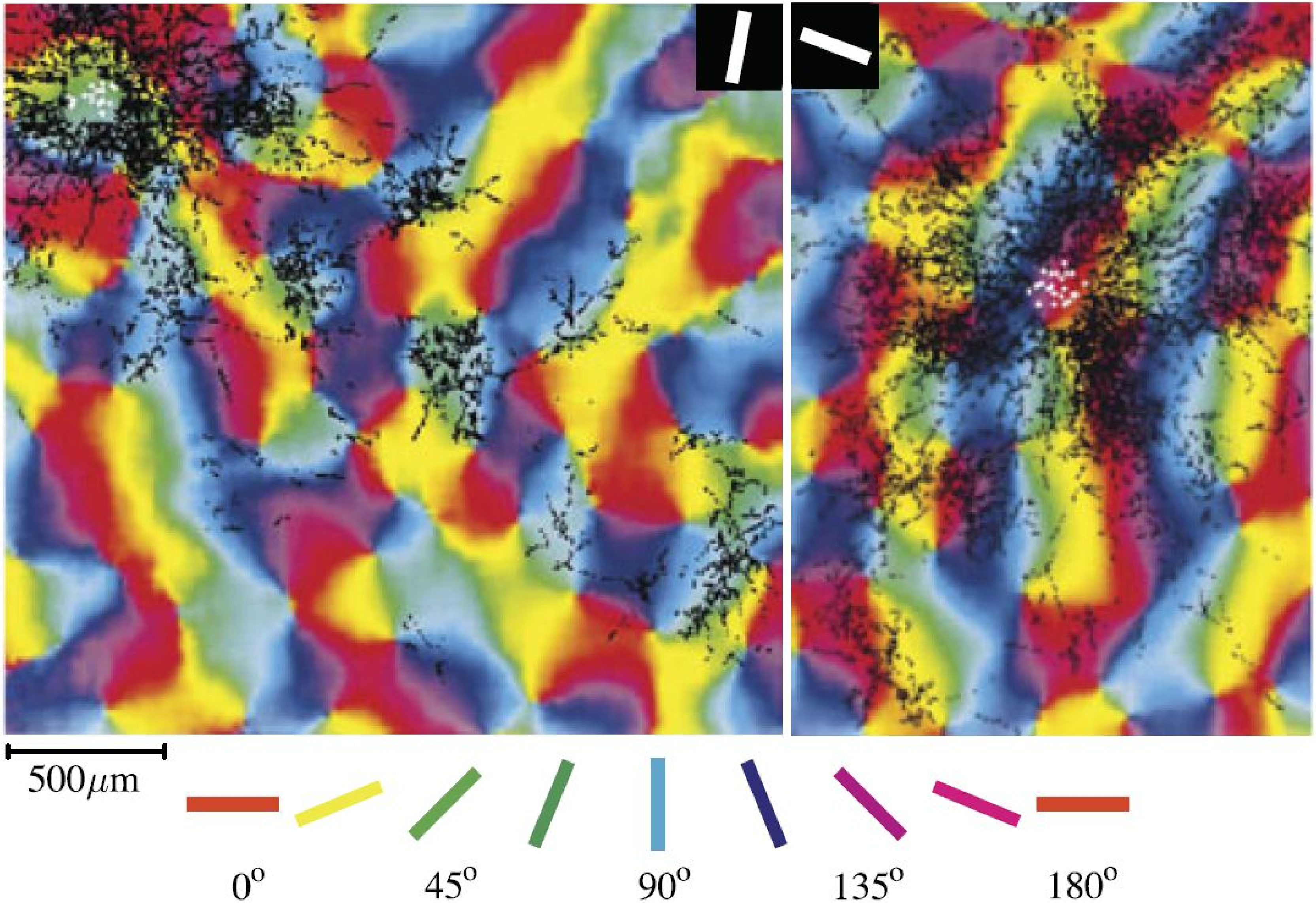

retina
optic
nerve
lense
artery
pupil
vienna
fovea


cones
rods
After Østerberg 1935
optic disc
Receptor Density [10³/mm²]
180
160
140
120
100
80
60
40
20
0
70 60 50 40 30 20 10 0 10 20 30 40 50 60 70 80 90
Temporal Retina
Eccentricity [°]
Nasal Retina

# Perception of Visual Information in Human Brain

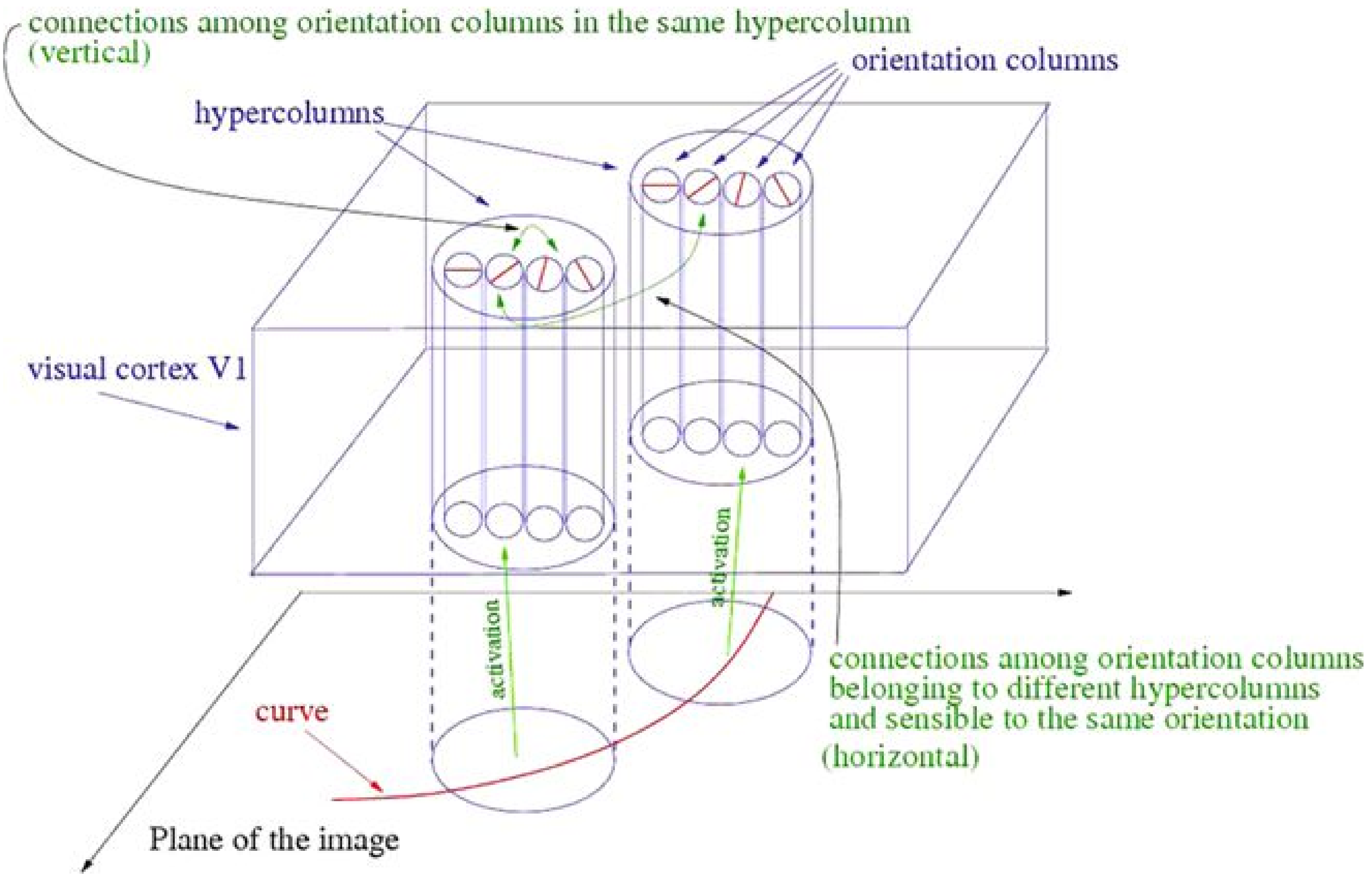


# Primary Visual Cortex V1




Replicated from Bosking, W.H., et al., Orientation selectivity and the arrangement of horizontal connections in tree shrew striate cortex. J. Neuroscience, 1997

# A Model of the Primary Visual Cortex V1

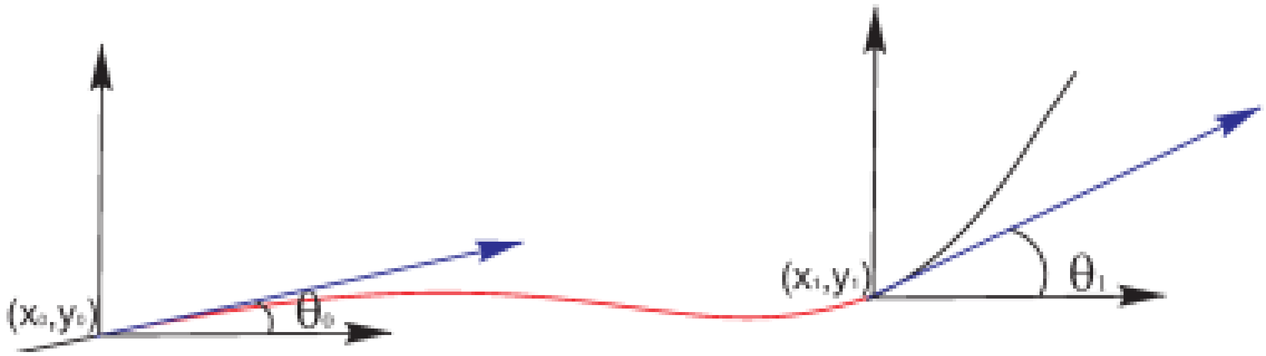



Replicated from R. Duits, U. Boscain, F. Rossi, Y. Sachkov,
Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2), JMIV, 2013.

# Cortical Based Model of Perceptual Completion

- D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.
- Sub-Riemanian structures in neurogeometry of the vision:
  - J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003. (Heisenberg group.)
  - G. Citti and A. Sarti, A Cortical Based Model of Perceptual Completion in the Roto-Translation Space, 2006. ($SE(2)$ group.)
- Variational principle: recovered arc has minimal length in the space $(x, y, \theta)$:

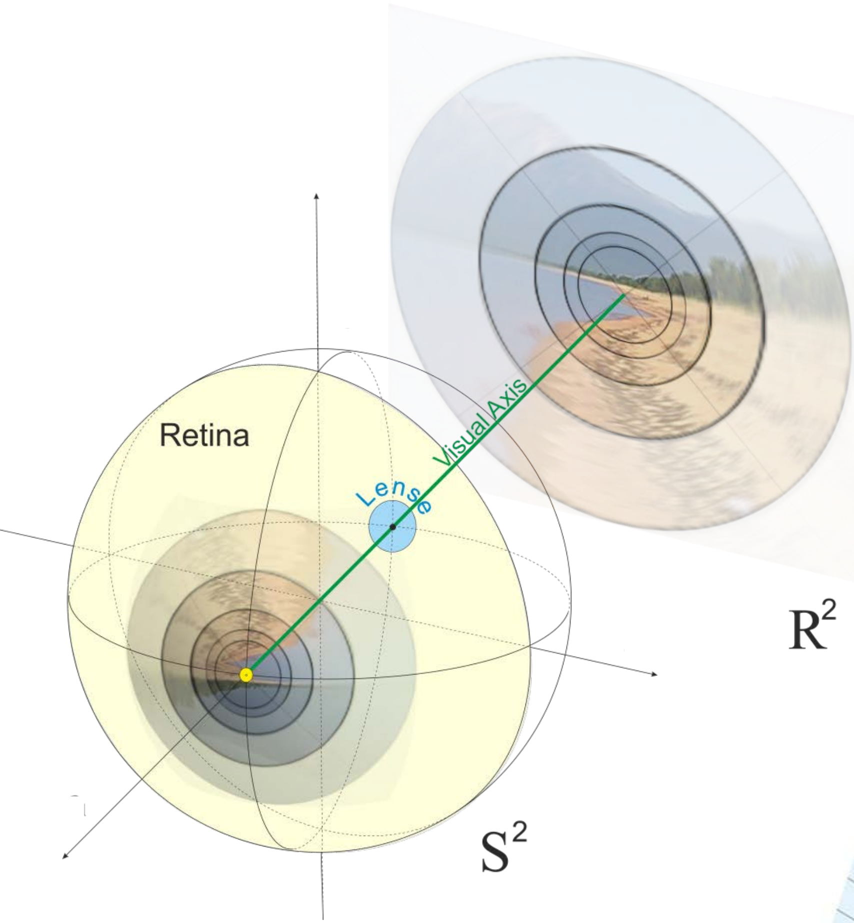
$$\int \sqrt{\xi^2 \left(\dot{x}^2 + \dot{y}^2\right) + \dot{\theta}^2} dt \to \min, \text{ under constraint } \dot{\theta} = \arg\left(\dot{x} + \mathrm{i}\ \dot{y}\right)$$



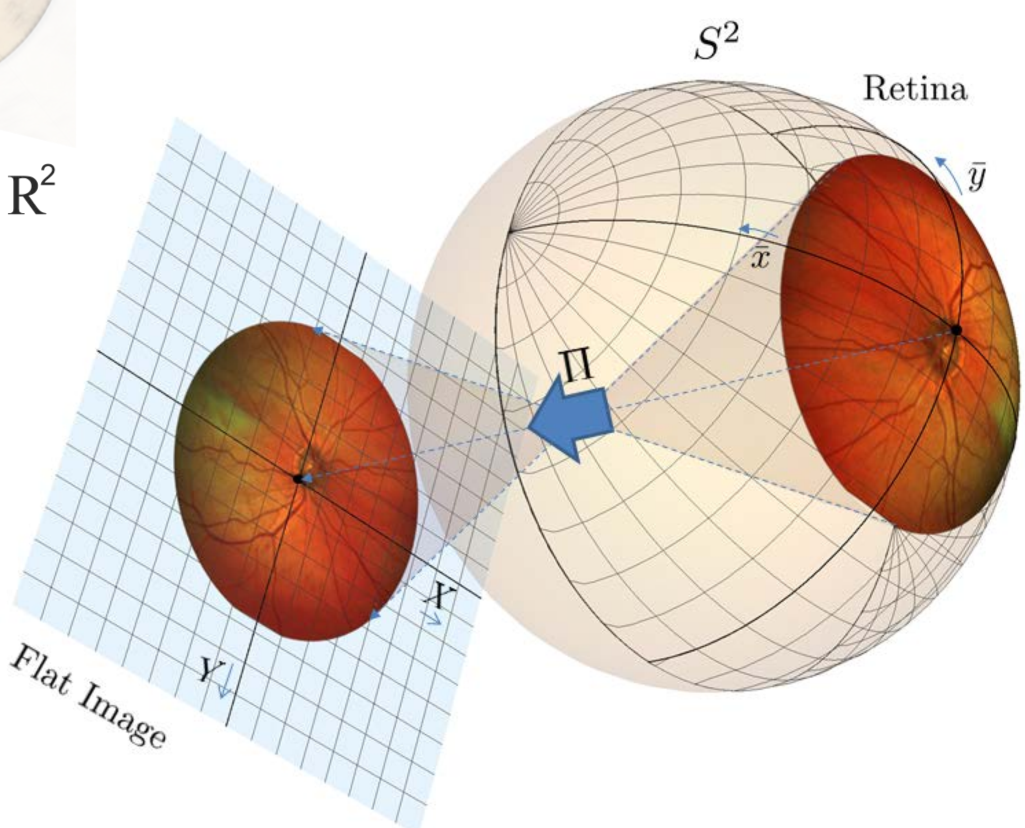
# Motivation

The retina is not flat

Nonuniform distribution of photoreceptors
=> Cortical magnification



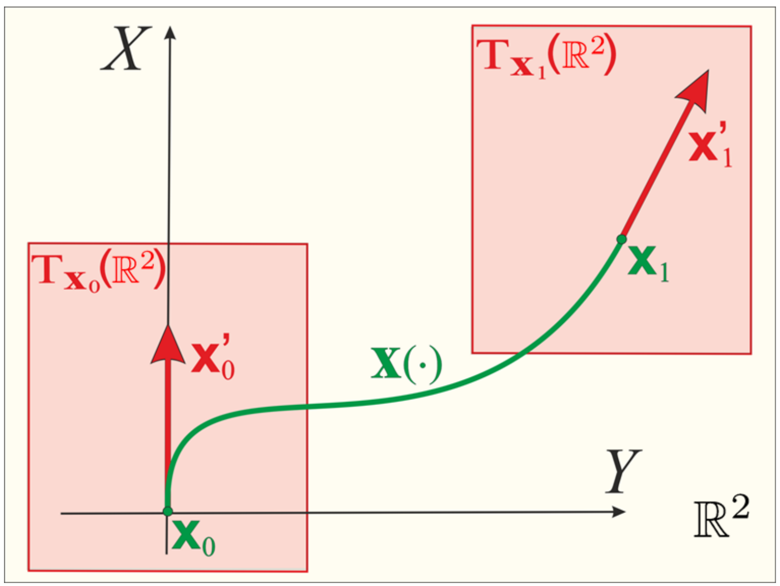

Spherical extension of cortical based model of perceptual completion on retinal sphere



# Spherical Extension of Flat Model

**Given:** constant $\xi > 0$,
$\mathbf{x}_0 \in \mathbb{R}^2$, $\mathbf{x}_0' \in T_{\mathbf{x}_0}(\mathbb{R}^2)$,
$\mathbf{x}_1 \in \mathbb{R}^2$, $\mathbf{x}_1' \in T_{\mathbf{x}_1}(\mathbb{R}^2)$,
external cost $\mathfrak{c} : \mathbb{R}^2 \to \mathbb{R}^+$

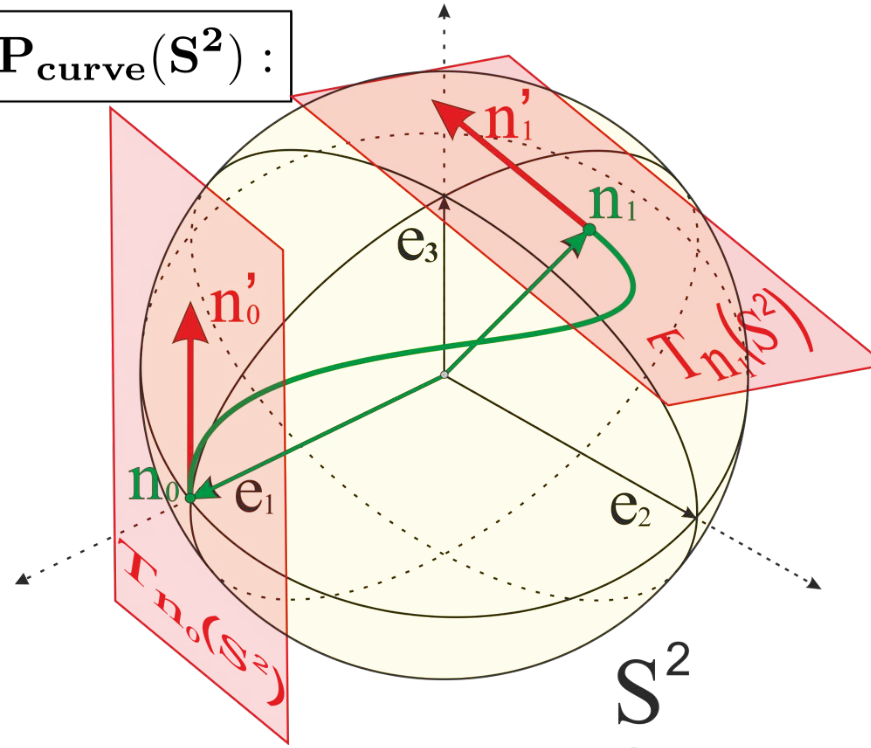
$\mathbf{P_{curve}}(\mathbb{R}^2)$ :




**Find:** $\mathbf{x}(\cdot) : [0, l] \to \mathbb{R}^2$, s.t.
$\mathbf{x}(0) = \mathbf{x}_0, \quad \mathbf{x}(l) = \mathbf{x}_1,$
$\mathbf{x}'(0) = \mathbf{x}_0', \quad \mathbf{x}'(l) = \mathbf{x}_1',$
$\int_0^l \mathfrak{c}(\mathbf{n}(s))\sqrt{\xi^2 + k^2(s)}\mathrm{d}s \to \min.$

**Given:** constant $\xi > 0$,
$\mathbf{n}_0 \in S^2$, $\mathbf{n}_0' \in T_{\mathbf{n}_0}(S^2)$,
$\mathbf{n}_1 \in S^2$, $\mathbf{n}_1' \in T_{\mathbf{n}_1}(S^2)$,
external cost $\mathfrak{C} : S^2 \to \mathbb{R}^+$

$\mathbf{P_{curve}}(\mathbf{S^2})$ :




**Find:** $\mathbf{n}(\cdot) : [0, l] \to S^2$, s.t.
$\mathbf{n}(0) = \mathbf{n}_0, \quad \mathbf{n}(l) = \mathbf{n}_1,$
$\mathbf{n}'(0) = \mathbf{n}_0', \quad \mathbf{n}'(l) = \mathbf{n}_1',$
$\int_0^l \mathfrak{C}(\mathbf{n}(s))\sqrt{\xi^2 + k_g^2(s)}\mathrm{d}s \to \min.$

# Sub-Riemannian Structure in SO(3)

Lie group $\mathrm{SO}(3) \ni g \sim R(x,y,\theta) = R^{y}_{\mathbf{e}_3} R^{-x}_{\mathbf{e}_2} R^{\theta}_{\mathbf{e}_1}$,
where $R^{\varphi}_{\mathbf{a}}$ is a 3D rotation around axis $\mathbf{a} \in S^2$ by angle $\varphi$.

Basis left-invariant vector fields

$$\begin{aligned}
\mathcal{A}_1|_g &= \cos\theta \, \partial_x|_g - \sec x \sin\theta \, \partial_y|_g + \tan x \sin\theta \, \partial_\theta|_g = (L_g)_* \, \partial_x|_e \,, \\
\mathcal{A}_2|_g &= \partial_\theta|_g = (L_g)_* \, \partial_\theta|_e \,, \\
\mathcal{A}_3|_g &= \sin\theta \, \partial_x|_g + \sec x \cos\theta \, \partial_y|_g - \tan x \cos\theta \, \partial_\theta|_g = (L_g)_* \, \partial_y|_e \,,
\end{aligned}$$

where $(L_g)_*$ is push-forward of left multiplication $L_g h = gh$.

Basis left-invariant one forms $\langle \omega^i, \mathcal{A}_j \rangle = \delta^j_i$

Left-invariant distribution $\Delta = \mathrm{span}\{\mathcal{A}_1, \mathcal{A}_2\} \subset T(\mathrm{SO}(3))$

Metric tensor $\mathcal{G}|_g = \mathcal{C}^2(g) \left( \xi^2 \omega^1 \otimes \omega^1 + \omega^2 \otimes \omega^2 \right)\big|_g$ on $\Delta$,

with external cost $\mathcal{C} : \mathrm{SO}(3) \to [\delta, +\infty)$, $\delta > 0$, and $\xi > 0$.

SR-distance: Inf among Lipschitzian curves $\gamma : [0,T] \to \mathrm{SO}(3)$

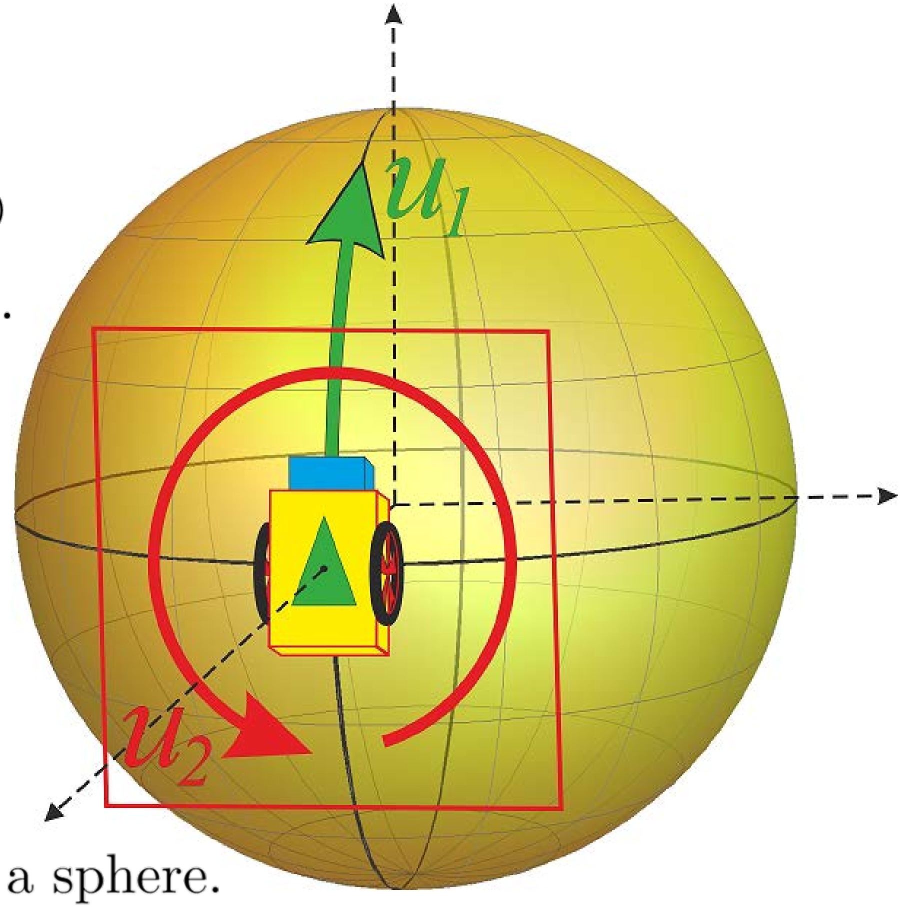
$$d(e,g) = \inf\{ \int_0^T \sqrt{\mathcal{G}|_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t))} \mathrm{d}\,t \mid \begin{array}{l} \gamma(0) = e, \\ \gamma(T) = g, \end{array} \; \dot\gamma(t) \in \Delta|_{\gamma(t)} \}.$$

# Mechanical Interpretation of Problem Pmec

$$\boxed{\mathbf{P_{mec}}:} \qquad \gamma(0) = e, \quad \gamma(T) = g,$$

$$\dot\gamma(t) = u_1(t)\,\mathcal{A}_1|_{\gamma(t)} + u_2(t)\,\mathcal{A}_2|_{\gamma(t)}$$

$$\int_0^T \mathcal{C}(\gamma(t))\sqrt{\xi^2 u_1(t)^2 + u_2(t)^2}\,\mathrm{d}\,t \to \min.$$




Optimal motion of Reeds-Shepp car on a sphere.

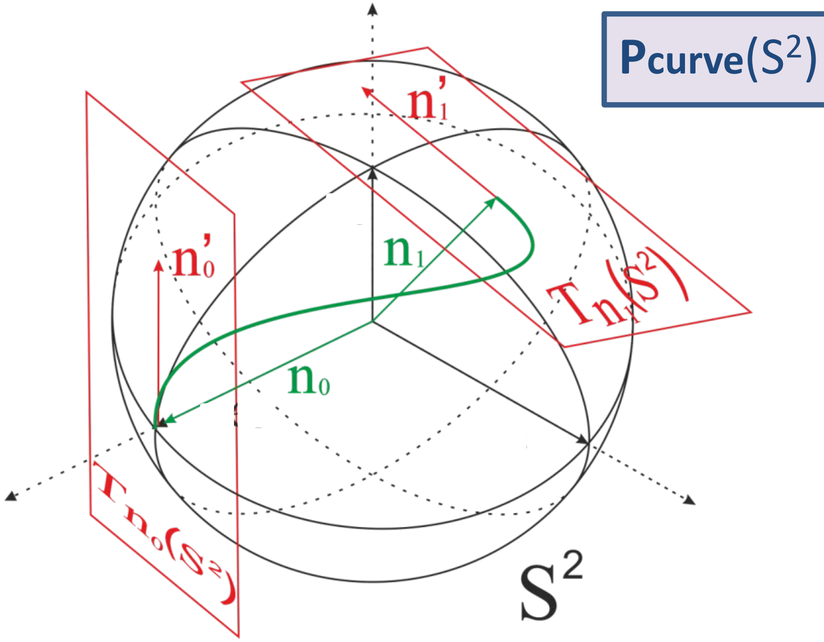
Admissible motions forward/backward and rotations on a place are controlled by $(u_1, u_2) \in \mathbb{R}^2$.

# Problem **Pcurve**($S^2$) and **Pmec**(SO(3))

**Given** $\xi > 0$,
$\mathbf{n}_0 \in S^2$, $\mathbf{n}_0' \in T_{\mathbf{n}_0}(S^2)$,
$\mathbf{n}_1 \in S^2$, $\mathbf{n}_1' \in T_{\mathbf{n}_1}(S^2)$,

**Find** a smooth curve $\mathbf{n}(\cdot) : [0, l] \to S^2$ s. t.:

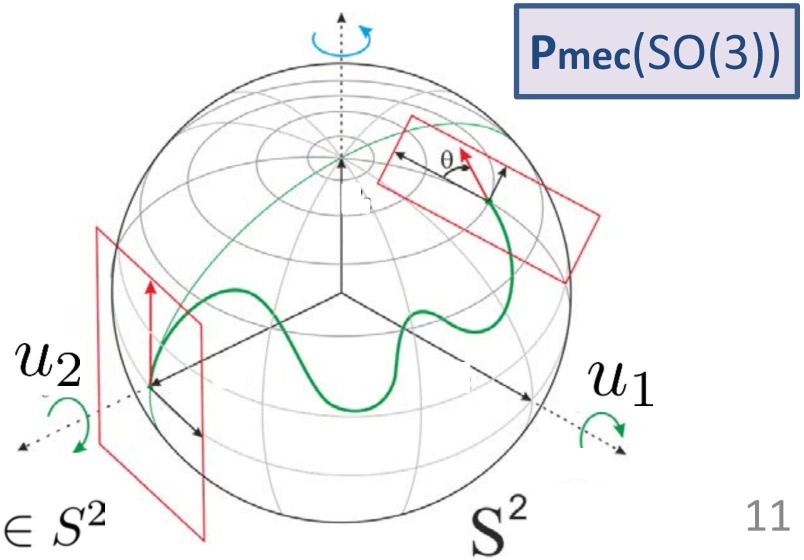
$$\mathbf{n}(0) = \mathbf{n}_0, \quad \mathbf{n}(l) = \mathbf{n}_1,$$
$$\mathbf{n}'(0) = \mathbf{n}_0', \quad \mathbf{n}'(l) = \mathbf{n}_1',$$
$$\int_0^l \mathfrak{C}(\mathbf{n}(s))\sqrt{\xi^2 + k_g^2(s)}\mathrm{d}s \to \min.$$




**Given** $\xi > 0$, $\quad g \in \mathrm{SO}(3)$

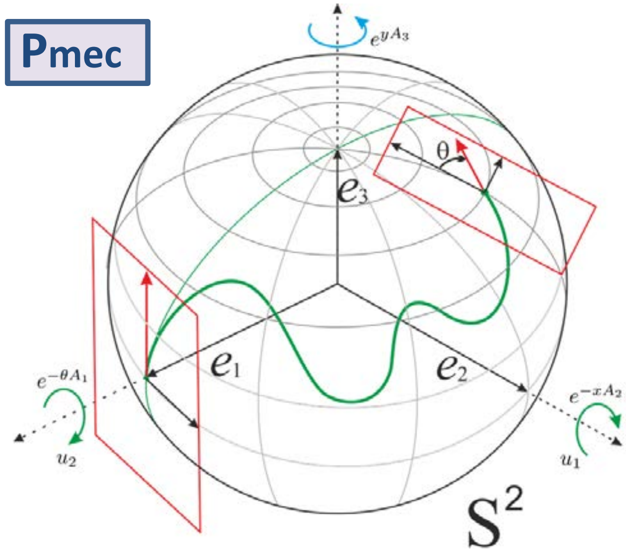
**Find** a trajectory $\gamma : [0, T] \to \mathrm{SO}(3)$ s.t.:

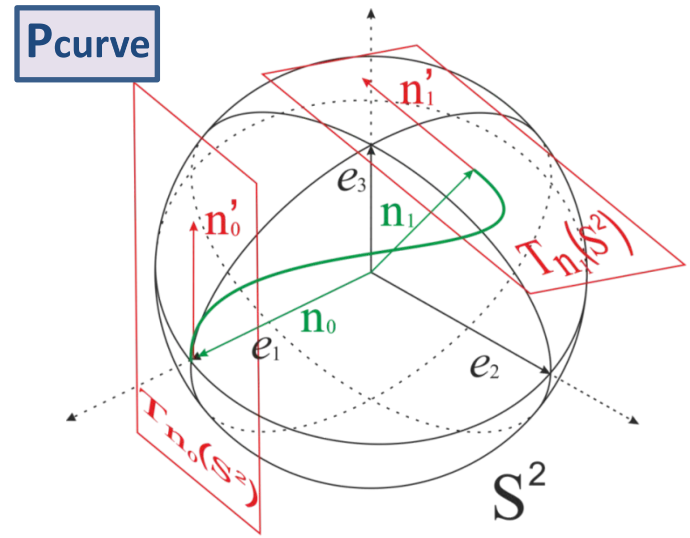
$$\gamma(0) = e, \quad \gamma(T) = g,$$
$$\dot{\gamma}(t) = u_1(t)\, \mathcal{A}_1|_{\gamma(t)} + u_2(t)\, \mathcal{A}_2|_{\gamma(t)}$$
$$\int_0^T \mathcal{C}(\gamma(t))\sqrt{\xi^2 u_1(t)^2 + u_2(t)^2}\, \mathrm{d}\, t \to \min.$$




Spherical projection: $\mathrm{SO}(3) \ni \gamma(t) \mapsto \gamma(t)e_1 \in S^2$

# Relation between Pmec and Pcurve




**Theorem.**

Let $\gamma(t)$, $t \in [0, T]$, be a minimizer of $\mathbf{P}_{\mathbf{mec}}$ parametrized by SR-arclength $t$.
Let $u_1(t) > 0$ for all $t \in [0, T]$.

Set $\mathbf{n}_0 = \mathbf{e}_1$, $\mathbf{n}_0' = \mathbf{e}_3$, $\mathbf{n}_1 = \gamma(T)\,\mathbf{e}_1$, $\mathbf{n}_1' = \gamma(T)\,\mathbf{e}_3$.

Then for such boundary conditions $\mathbf{P}_{\mathbf{curve}}$ has a minimizer $\mathbf{n}(s)$,
along which

$$\mathbf{n}(s) = \gamma(t(s))\,\mathbf{e}_1, \qquad u_1(t) = \frac{ds}{dt}(t), \;\; u_2(t) = k_g(s(t))\frac{ds}{dt}(t),$$

and $t(s) = \int_0^s \mathfrak{C}(\mathbf{n}(\sigma))\sqrt{\xi^2 + k_g^2(\sigma)}\,d\sigma.$

# Hamiltonian System of PMP for Pmec

By Cauchy-Schwartz $J = \frac{1}{2}\int_0^T \mathcal{C}^2(\gamma(t))(\xi^2 u_1^2(t) + u_2^2(t))\,dt \to \min,$

Basis left-invariant Hamiltonians $h_i = \langle \lambda, \mathcal{A}_i \rangle,\ \lambda \in T_g^*(\mathrm{SO}(3))$

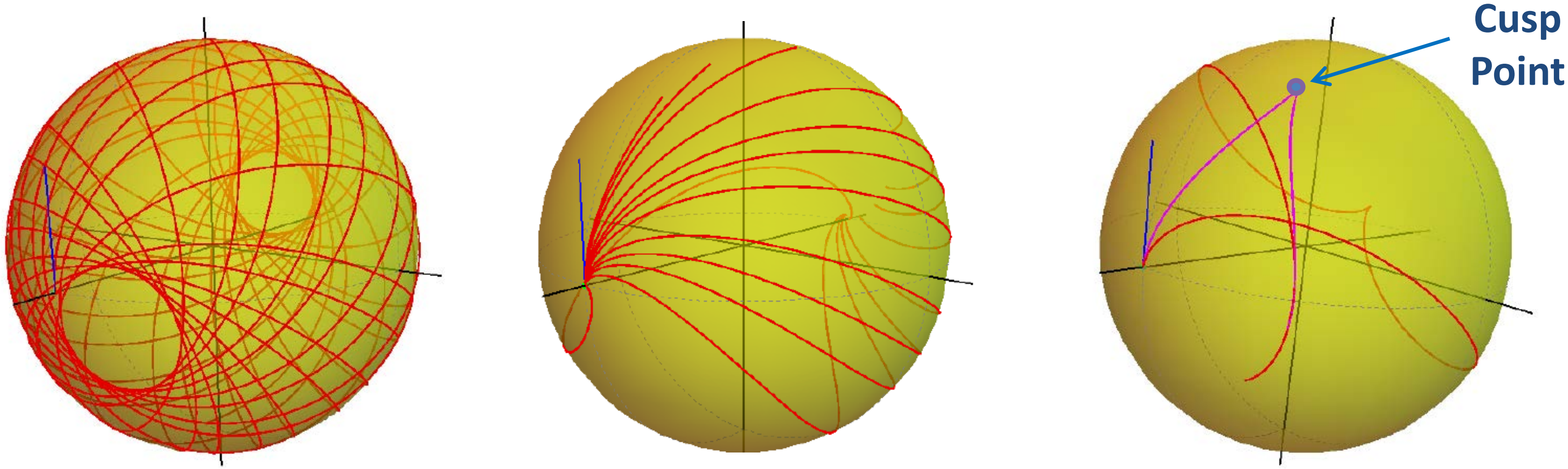
The (maximized) Hamiltonian $H = \frac{1}{2\,\mathcal{C}(g)^2}\left(\frac{h_1^2}{\xi^2} + h_2^2\right)$

$$\begin{cases} \dot h_1 = \frac{\mathcal{A}_1(\mathcal{C})}{\mathcal{C}} - \frac{h_2 h_3}{\mathcal{C}^2}, \\ \dot h_2 = \frac{\mathcal{A}_2(\mathcal{C})}{\mathcal{C}} + \frac{1}{\xi^2}\,\frac{h_1 h_3}{\mathcal{C}^2}, \\ \dot h_3 = \frac{\mathcal{A}_3(\mathcal{C})}{\mathcal{C}} + \left(1 - \frac{1}{\xi^2}\right)\frac{h_1 h_2}{\mathcal{C}^2} \end{cases}$$

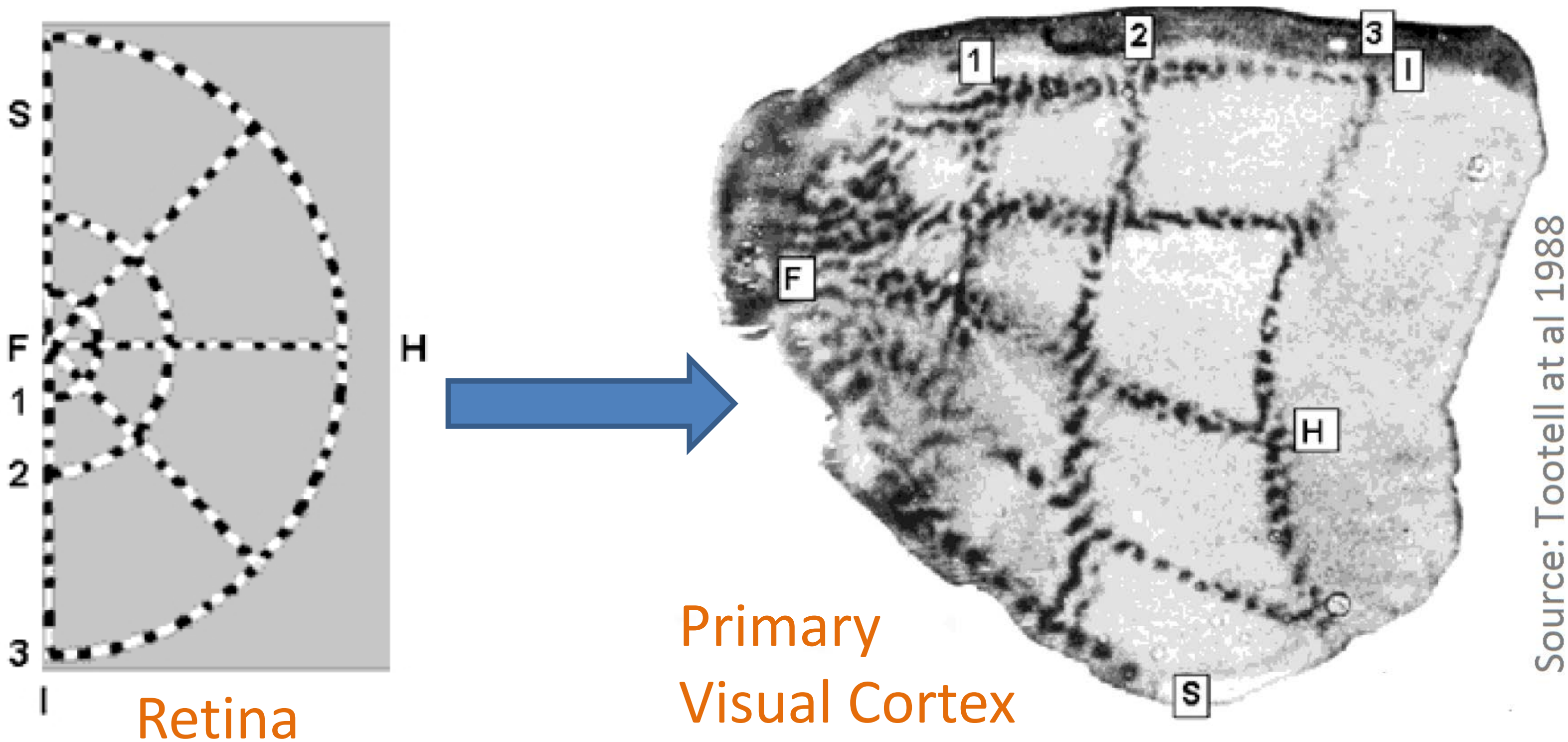
— vertical part,

$$\begin{cases} \dot x = \frac{h_1}{\xi^2 \mathcal{C}^2}\cos\theta, \\ \dot y = -\frac{h_1}{\xi^2 \mathcal{C}^2}\sec x \sin\theta, \\ \dot\theta = \frac{h_1}{\xi^2 \mathcal{C}^2}\sin\theta \tan x + \frac{h_2}{\mathcal{C}^2} \end{cases}$$

— horizontal part,



# Cortical Magnification




R.B.H. Tootell, E. Switkes, M.S. Silverman, S.J. Hamilton. Functional anatomy of macaque striate cortex. II. Retinotopic organization. Journal of Neuroscience, 1988.

# Modelling of Cortical Magnification

- L. Florack. Canonical Coordinates for Retino-Cortical Magnification. Axioms, 2014.

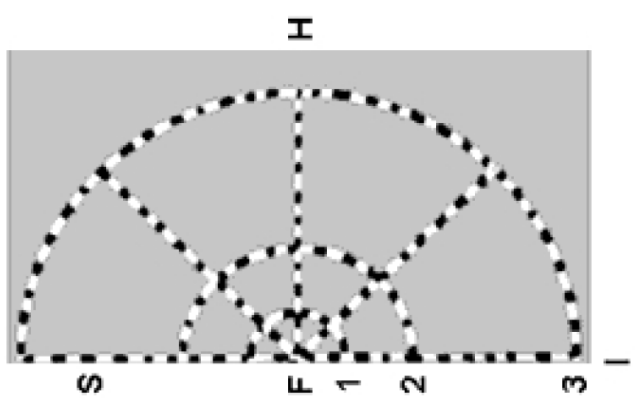
- Coordinate chart on the retinal hemisphere

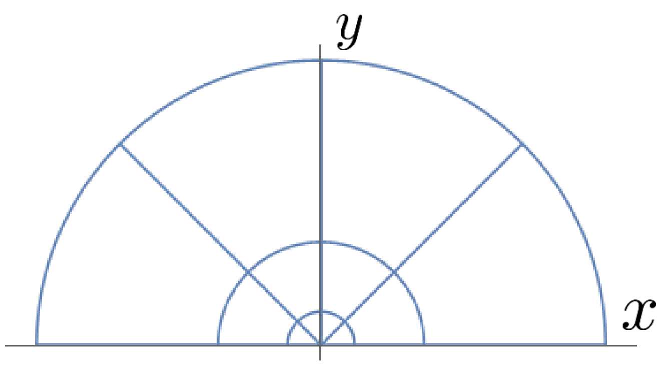
$$x = \sinh(p)\cos\phi, \quad y = \sinh(p)\sin\phi.$$

- The integrated retino-cortical magnification

$$v(x,y) = \frac{\log(1 + (x^2 + y^2)/a^2)}{\log(1 + T_0^2)},$$

$a \approx 0.015$ represents a transient radius separating the geometric foveola, $T_0 \approx 95$ the maximal size of retinal region involved in visual perception.

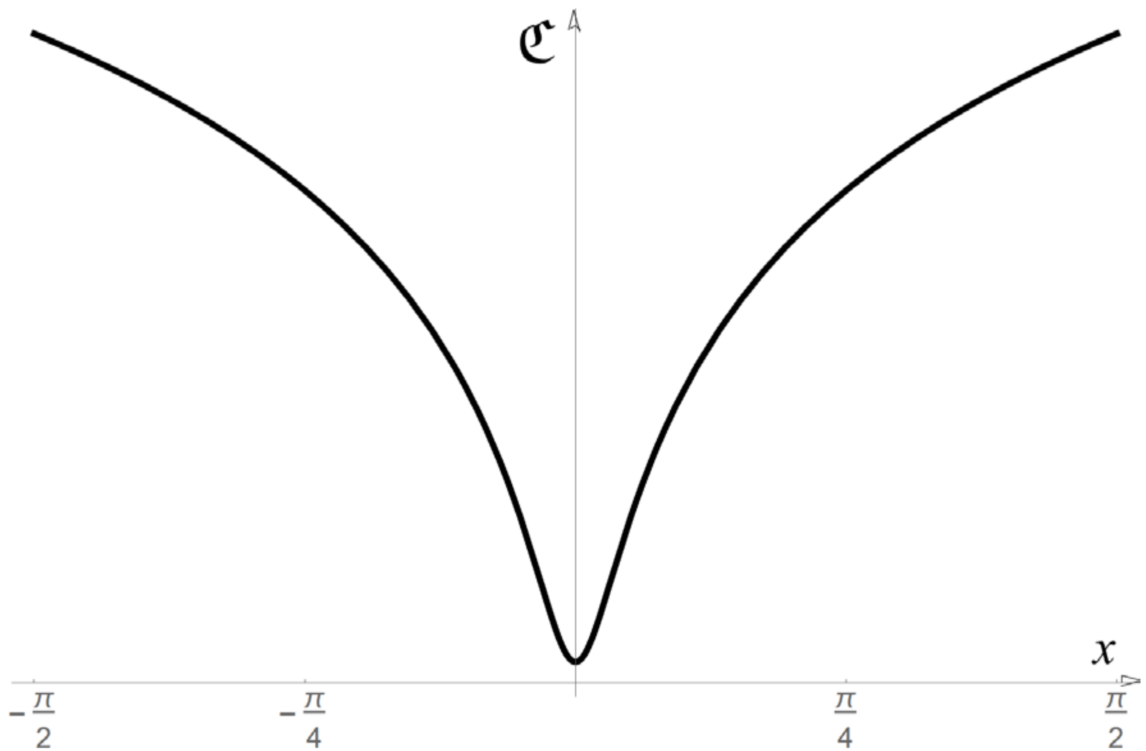


# Construction of the External Cost

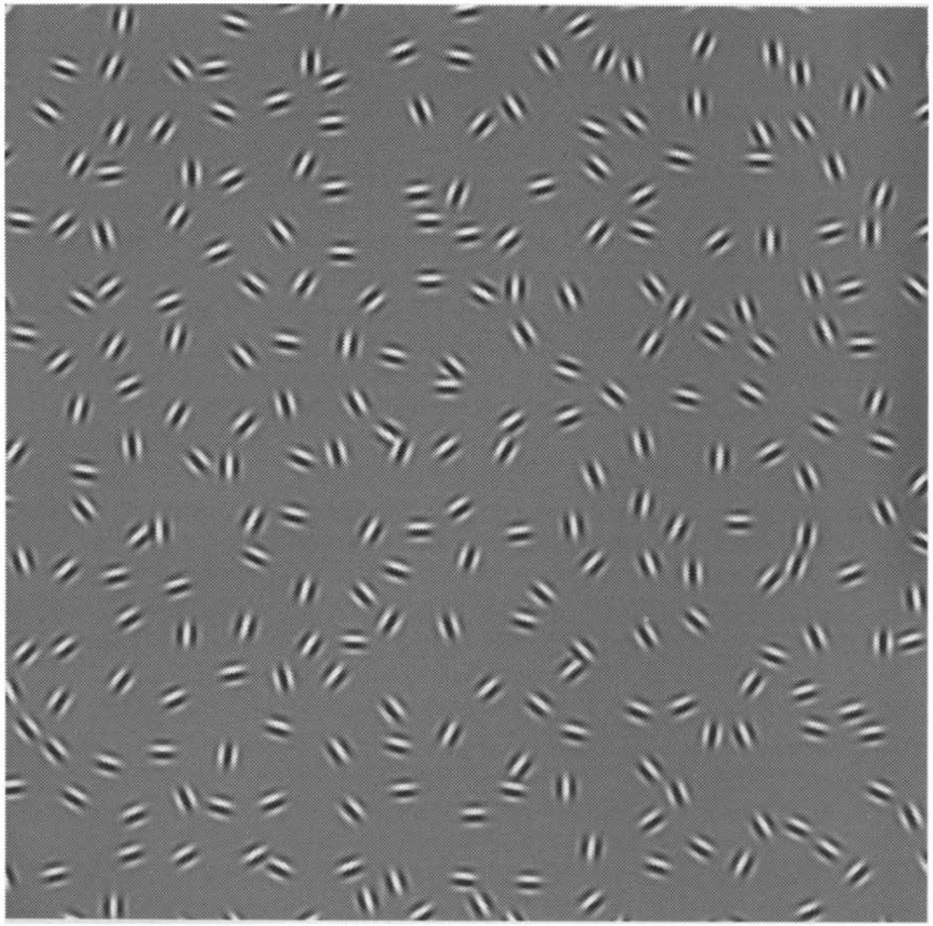
External cost $\mathfrak{C}(x, y)$ does not put a penalty in the foveola $\mathfrak{C}(0, 0) = 1$, and it penalizes a motion outside the foveola respectively to the cortical magnification

$$\mathcal{C}(x, y, \theta) = \mathfrak{C}(x, y) = 1 + v(x, y)$$

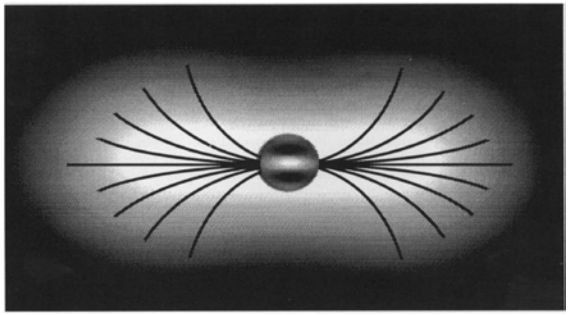


# Simulation of the Association Field

- D.J. Field, A. Hayes, R. Hess. Contour integration by the human visual system: Evidence for a local "association field", Vision Research, 1993.



Source: Field et al 1993



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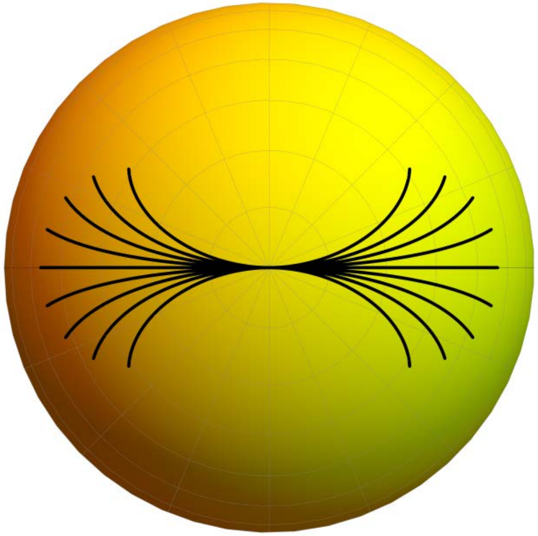

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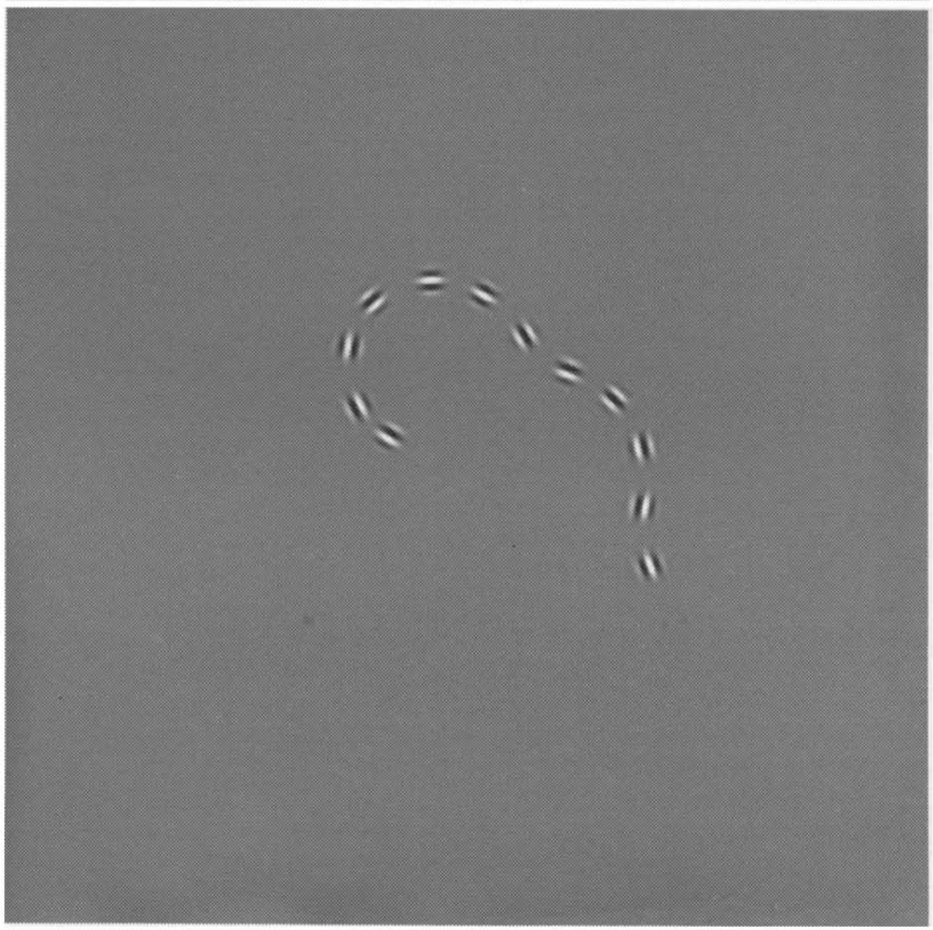

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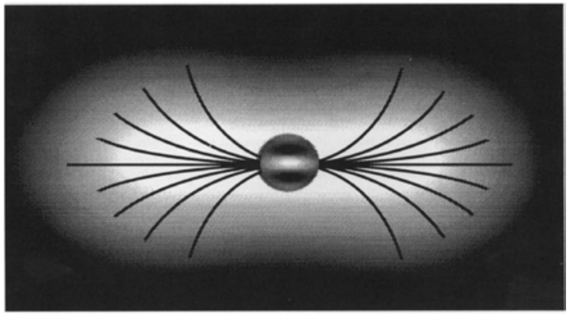

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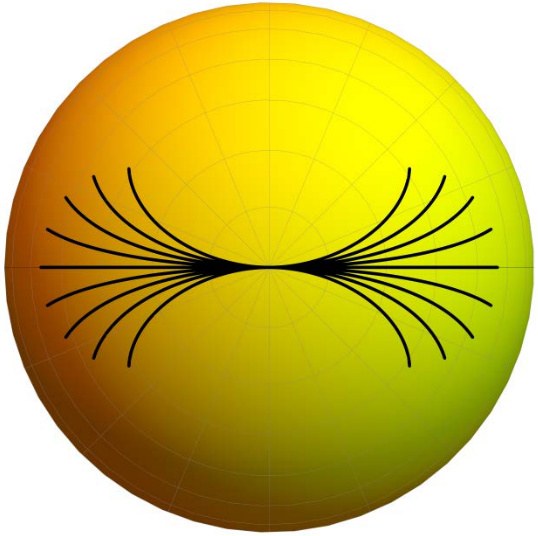

# Criteria of good continuation

Given a boundary condition $g_1 = (x_1, y_1, \theta_1)$ is perceptually connected to $e = (0, 0, 0)$, if the sub-Riemannian distance

$$d(e, g_1) \leq T$$

for some fixed $T > 0$.

Here, we also assume that $g_1$ is chosen such that it can be connected to $e$ with a minimizing geodesic whose spherical projection does not have a cusp.

# Conclusion

- cortical based model accounting spherical nature of the retina
- accounting nonuniform distributions of photoreceptors
- realistic external cost based on model of cortical magnification
- novel mathematical formulation of good continuation
- close approximation of the association field

# Continuation

- reconstruction of partially corrupted contours in images
- automatic training of the parameters
- Modelling of visual illusions

# Thank you for your attention!