Sub-Riemannian Geodesics on the Group of Motions of Euclidean Space

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Based on joint works with
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Motivation:

Detection of salient curves in images

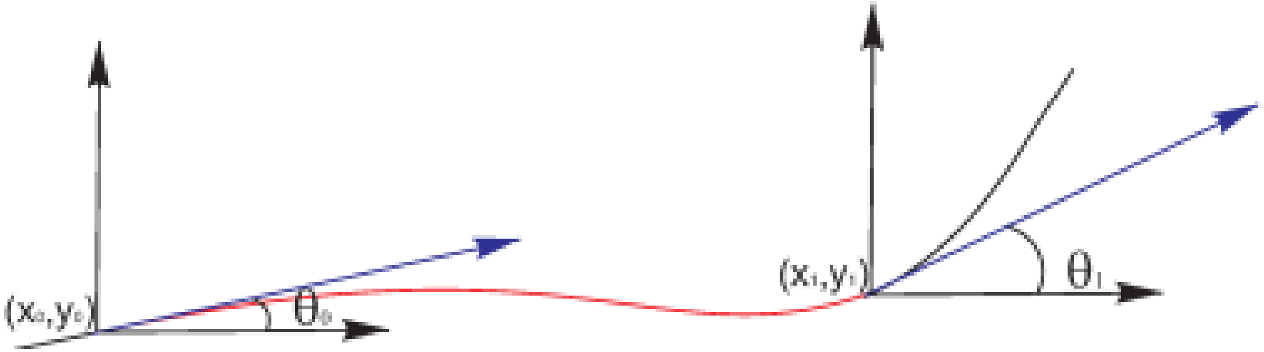
Cortical Based Model of Perceptual Completion

- D.H. Hubel and T.N. Wiesel, Receptive fields of single neurones in the cat's striate cortex, 1959. Nobel prize in 1981.

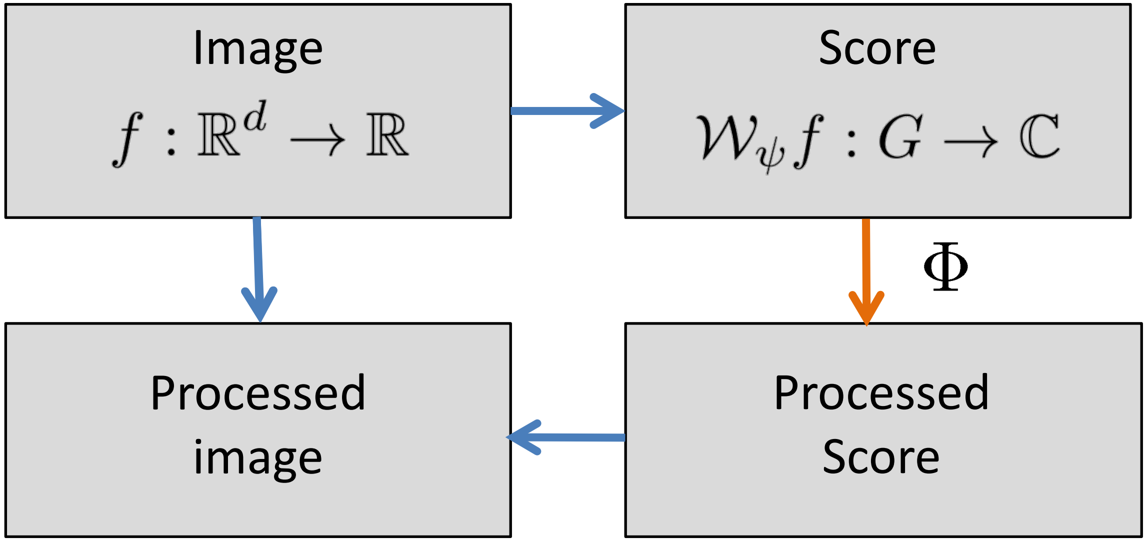
- Sub-Riemanian structures in neurogeometry of the vision:
  - J. Petitot, The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003. (Heisenberg group.)
  - G. Citti and A. Sarti, A Cortical Based Model of Perceptual Completion in the Roto-Translation Space, 2006. ($SE(2)$ group.)

- Variational principle: recovered arc has minimal length in the space (x, y, θ):

$$\int \sqrt{\xi^2\left(\dot{x}^2+\dot{y}^2\right)+\dot{\theta}^2 dt} \rightarrow \min, \text{ under constraint } \dot{\theta} = \arg\left(\dot{x} + \mathrm{i}\ \dot{y}\right)$$



Lie Group Analysis via Invertible Orientation Scores



Image

$f : \mathbb{R}^d \to \mathbb{R}$

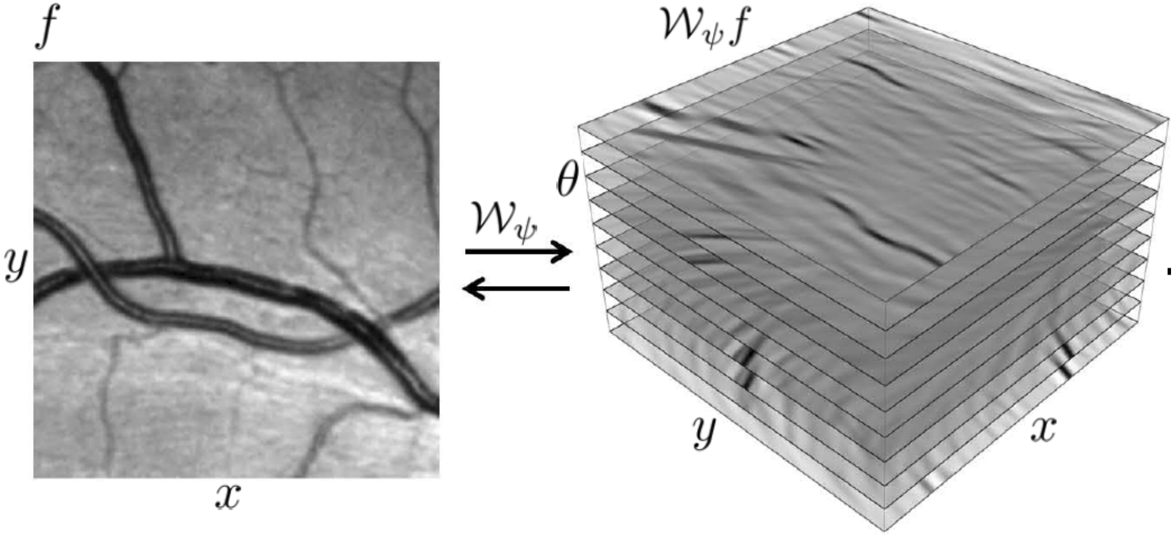
Score

$\mathcal{W}_\psi f : G \to \mathbb{C}$

Φ

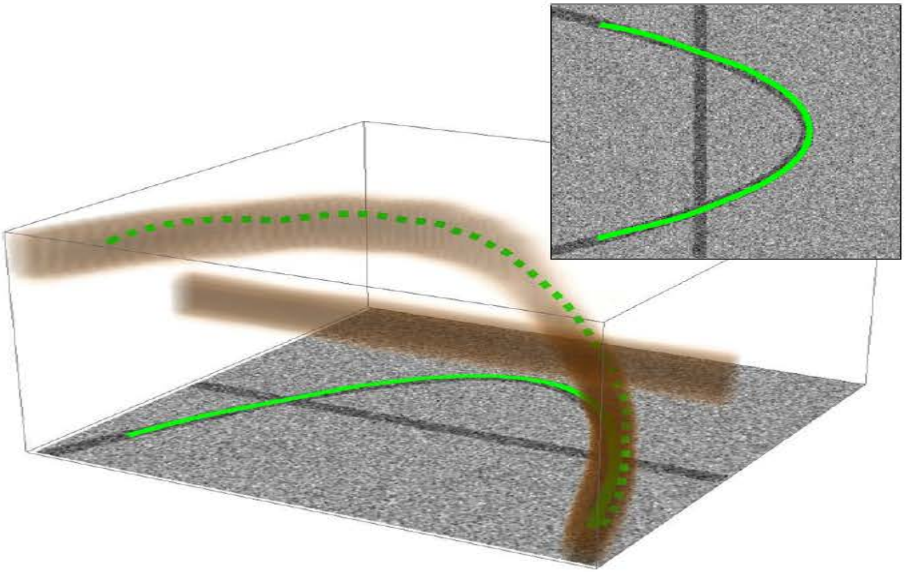
Processed image

Processed Score

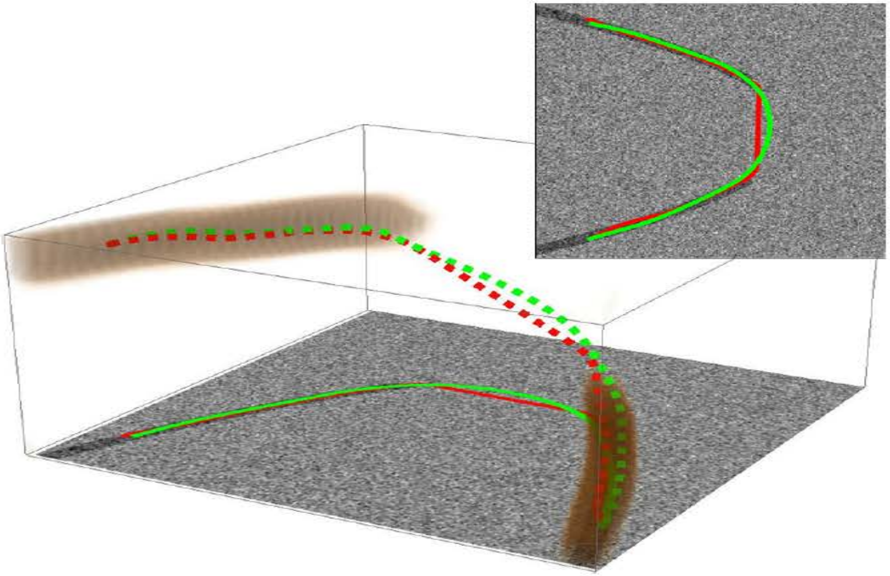


R. Duits: generic mathematical model for contextual image analysis via scores on Lie groups with many applications.

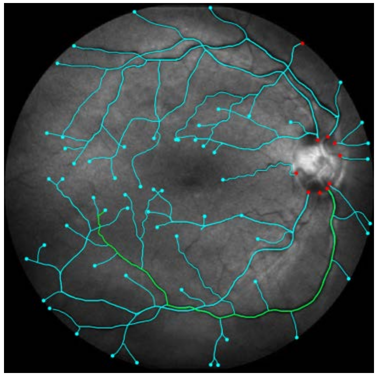
SR geodesics on Lie Groups in Image Analysis



Crossing structures are disentangled

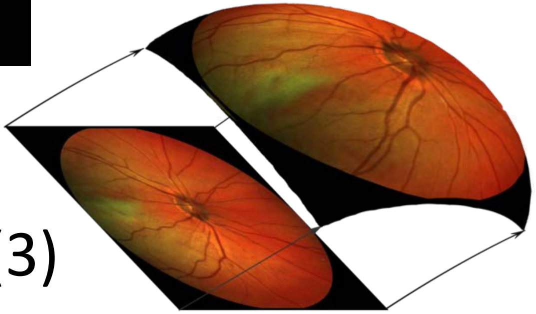


Restoration of corrupted contours based on model of human vision

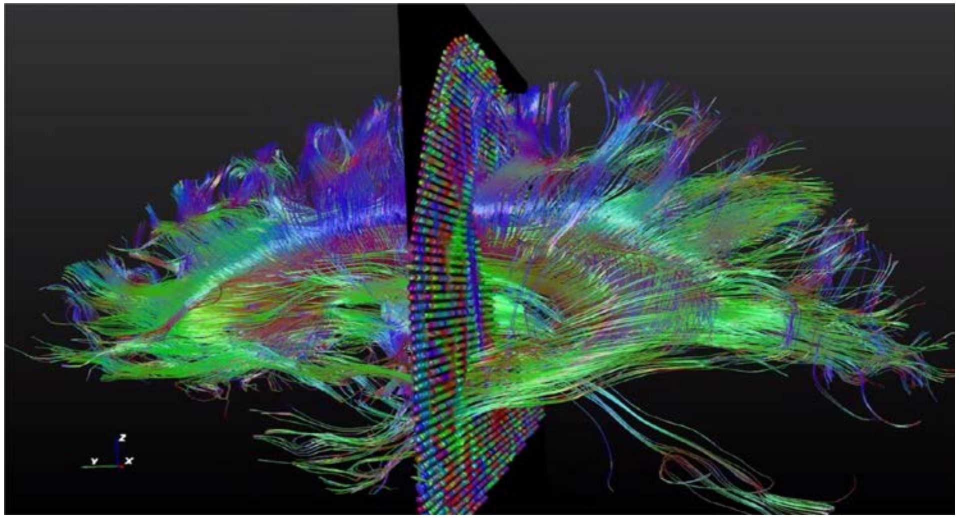


SE(2)

Applications in medical image analysis

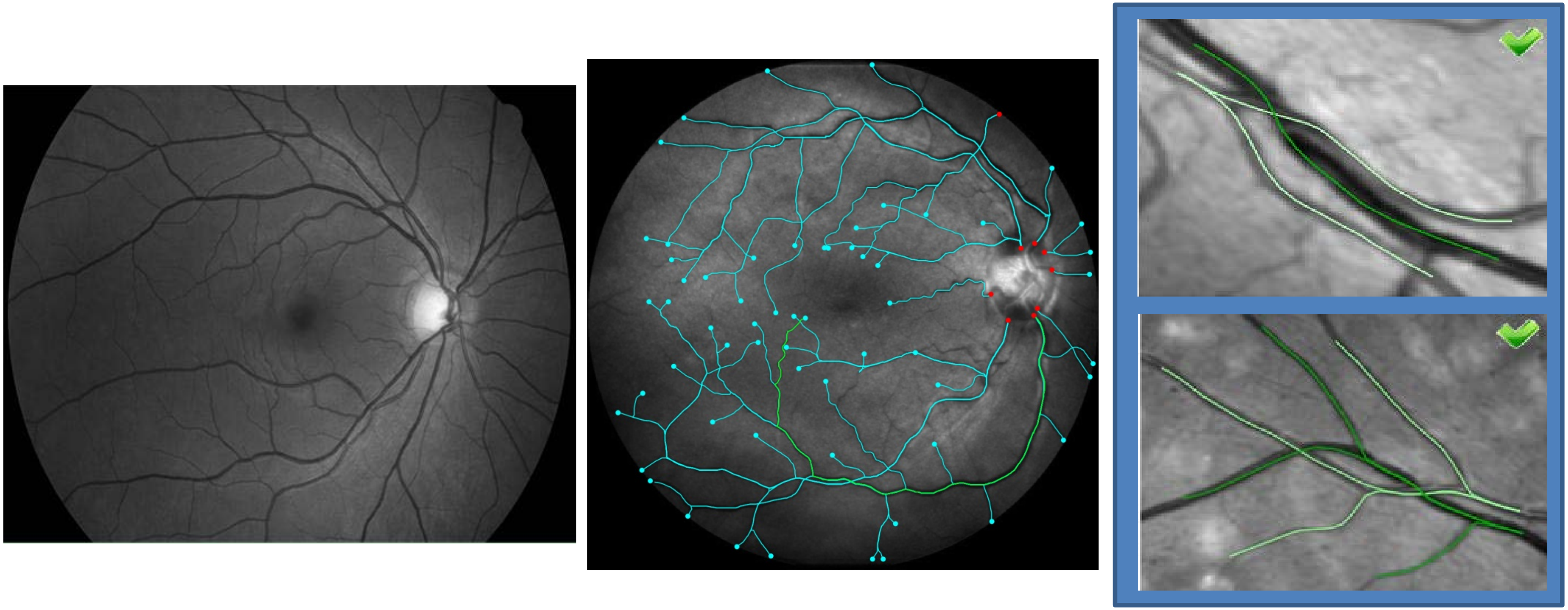


SO(3)



SE(3)

Tracking of Lines in Flat Images via Sub-Riemannian Geodesics in SE(2)

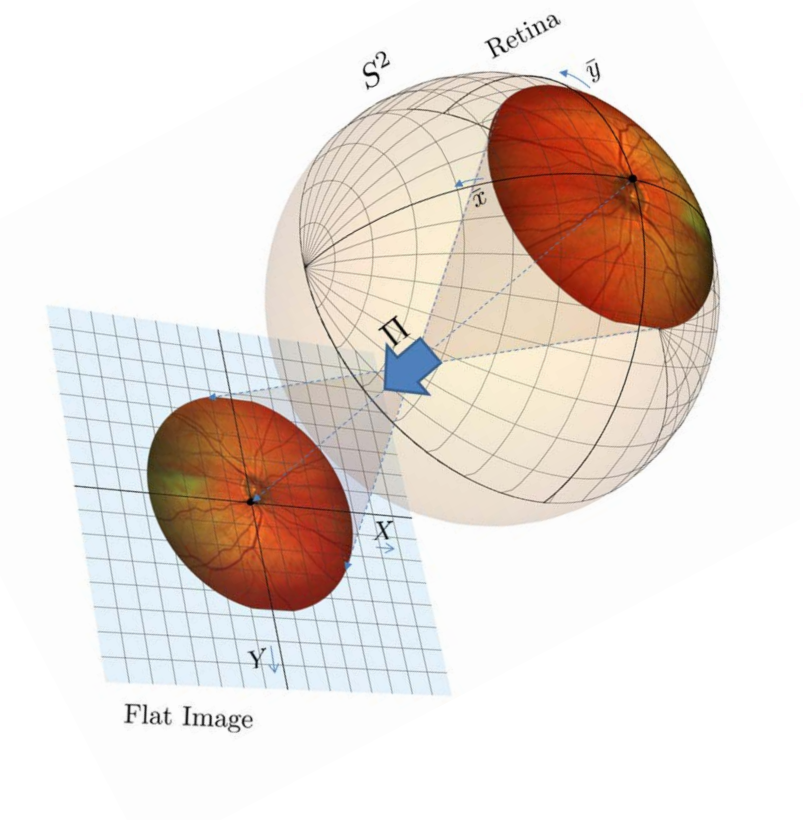


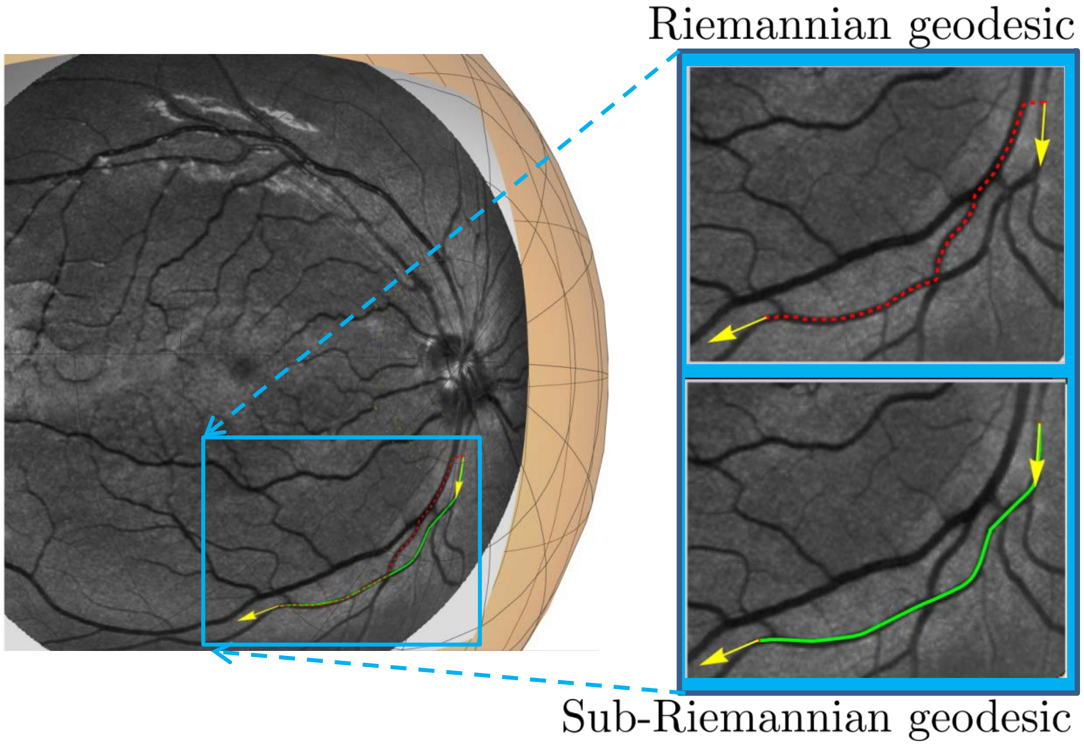
[**1**] E.J. Bekkers, R. Duits, A. Mashtakov and G.R. Sanguinetti,
Data-driven Sub-Riemannian Geodesics in SE(2), Proc. SSVM, 2015.

[**2**] E.J. Bekkers, R. Duits, A. Mashtakov and G.R. Sanguinetti,
A PDE Approach to Data-driven Sub-Riemannian Geodesics in SE(2), SIIMS, 2015.

[**3**] G. Sanguinetti, R. Duits, E. Bekkers, M. Janssen, A. Mashtakov, J-M. Mirebeau,
Sub-Riemannian Fast Marching in SE(2), Proc. CIARP, 2015.

Tracking of Lines in Spherical Images via Sub-Riemannian Geodesics in SO(3)



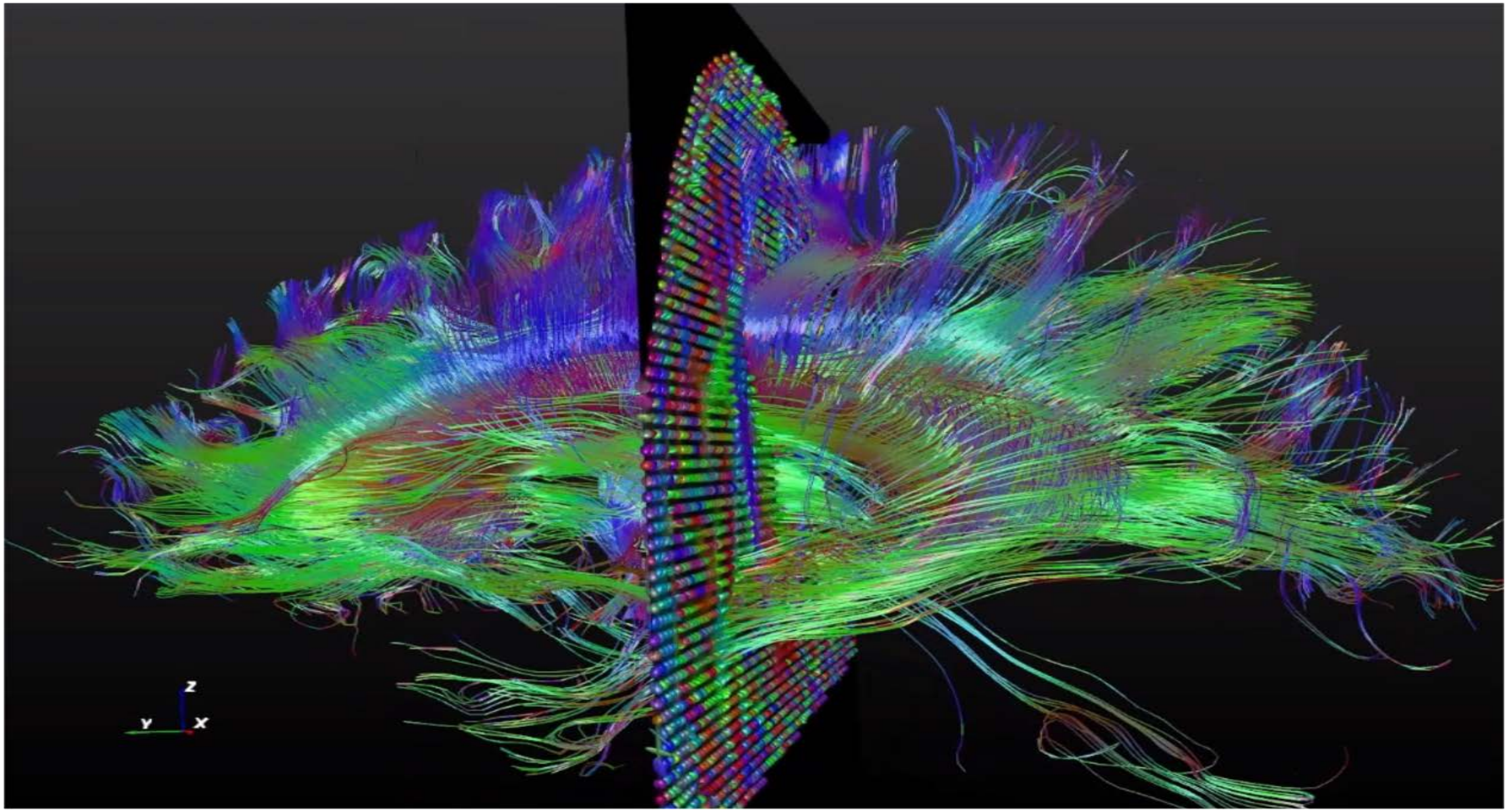


[**1**] A. Mashtakov, R. Duits, Yu. Sachkov, E.J. Bekkers, I. Beschastnyi, *Tracking of Lines in Spherical Images via Sub-Riemannian Geodesics in SO(3)*, JMIV, 2017.

[**2**] A.P. Mashtakov, R. Duits, Yu.L. Sachkov, E.J. Bekkers, I.Yu. Beschasnyi, *Sub-Riemannian Geodesics in SO(3) with Application to Vessel Tracking in Spherical Images of Retina*, Doklady Mathematics, 2017.

Sub-Riemannian Geodesics on **SE(3)**

Data-driven sub-Riemannian geodesics on **SE(3)** are used for detection and analysis of neuron fibers in magnetic resonance images of a human brain.



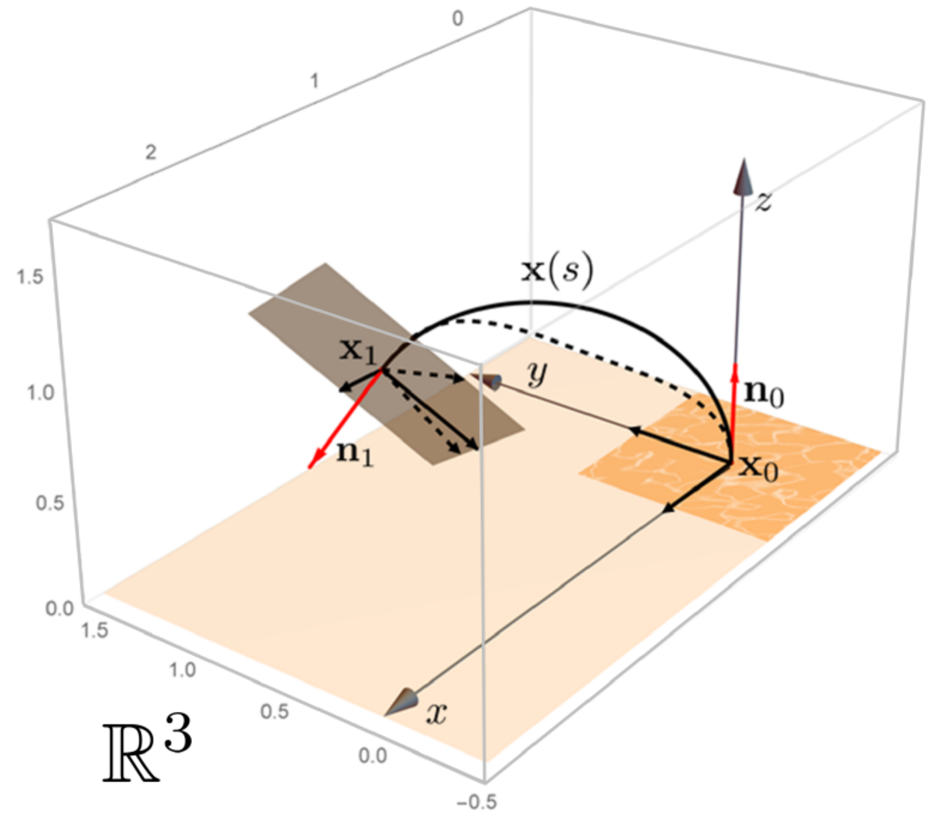
Problem **Pcurve($\mathbb{R}^3$)**: Shortest Path on $\mathbb{R}^3 \times S^2$

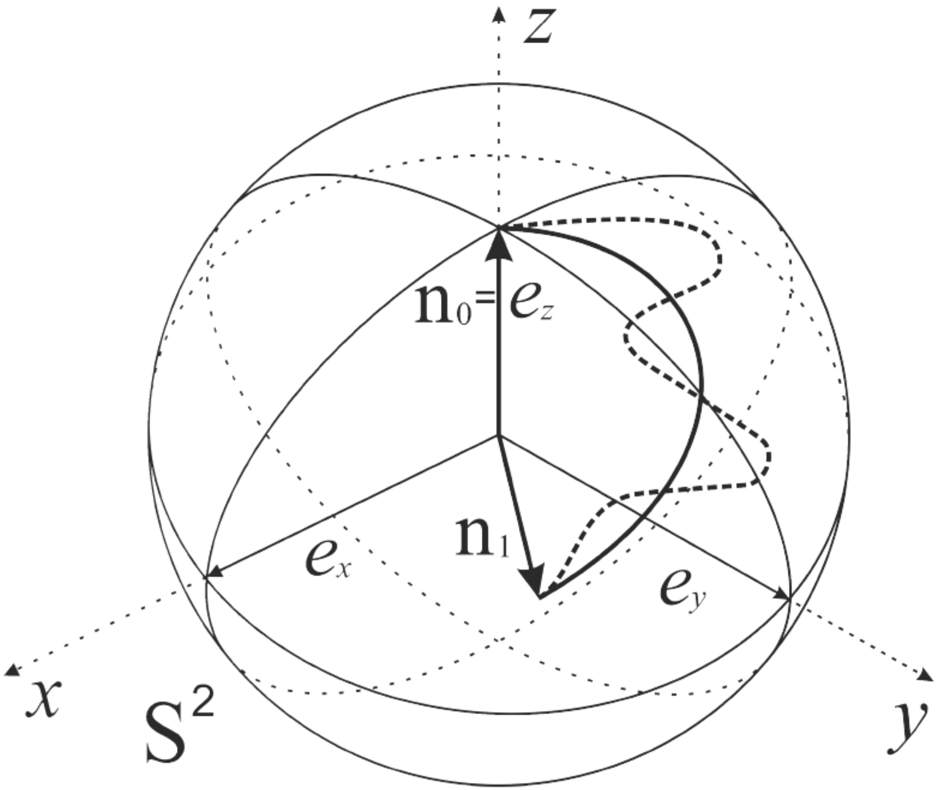
Given $\xi > 0, \quad \mathbf{x}_i \in \mathbb{R}^3, \;\; \mathbf{n}_i \in S^2, \;\; i \in \{0,1\}.$

Find a smooth curve $\mathbf{x} \in C^\infty([0,L], \mathbb{R}^3)$ s.t. $\mathbf{x}(0) = \mathbf{x}_0, \; \mathbf{x}(L) = \mathbf{x}_1 \in \mathbb{R}^3,$

$\mathbf{x}'(0) = \mathbf{n}_0, \; \mathbf{x}'(L) = \mathbf{n}_1 \in \mathrm{S}^2,$

and $E(\mathbf{x}) := \int_0^L \sqrt{\xi^2 + \kappa^2(s)}\, \mathrm{d}s \to \min,$ where $\kappa(s) = \|\mathbf{x}''(s)\|.$





Sub-Riemannian problem on SE(3)

Lie group SE(3)

The group of Euclidean motions of 3-dimensional space

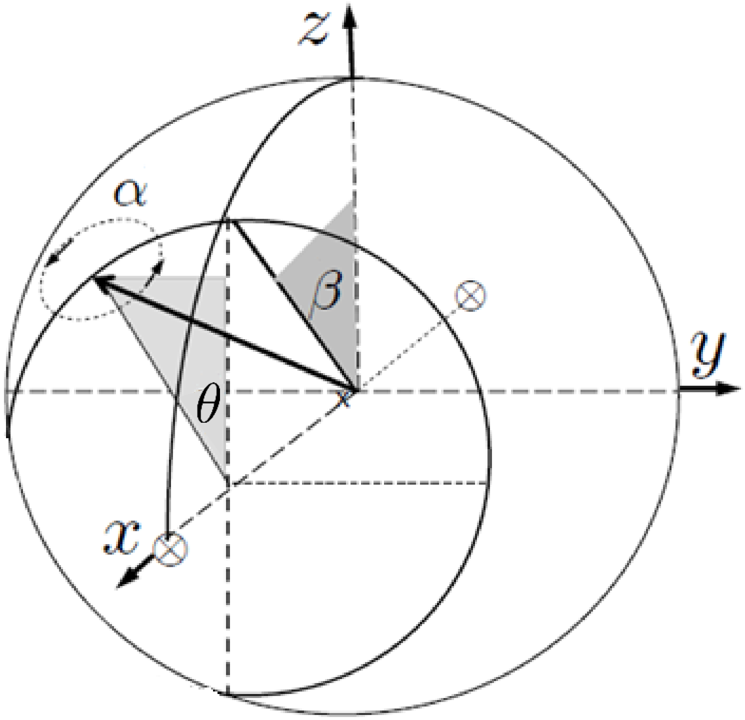
$$g = (\mathbf{x}, R) \in \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$$

Group operations

$$\begin{aligned} g_1 g_2 &= (\mathbf{x}_1, R_1)(\mathbf{x}_2, R_2) \\ &= (\mathbf{x}_1 + R_1\mathbf{x}_2, R_1 R_2), \\ g^{-1} &= (-R^T\mathbf{x}, R^T). \end{aligned}$$

We use the parameterization of SE(3)

$$\mathbf{x} = (x, y, z) \in \mathbb{R}^3,$$



$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\text{where } \alpha \in (-\pi, \pi], \quad \beta \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}], \quad \theta \in (-\pi, \pi]$$

Left-invariant Vector Fields on SE(3)

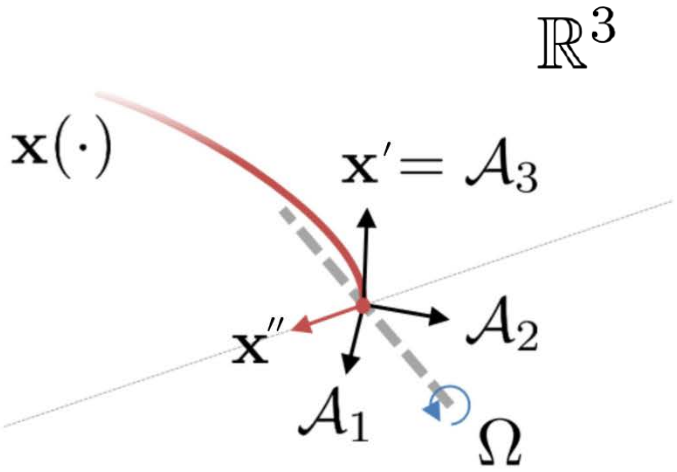
$$(\mathbf{x}, R) \in \mathrm{SE}(3) = \mathbb{R}^3 \rtimes \mathrm{SO}(3)$$

$$\mathrm{se}(3) = T_e\,\mathrm{SE}(3) = \mathrm{span}\{A_1, A_2, A_3, A_4, A_5, A_6\}$$

$$\mathcal{A}_i|_g = (L_g)_* A_i, \quad i \in \{1, \ldots, 6\}, \quad L_g h = gh$$

Co-frame $\{\omega^1, \ldots, \omega^6\}$: $\langle \omega^i, \mathcal{A}_j \rangle = \delta^i_j, \quad i, j \in \{1, \ldots, 6\}.$

$$\mathbb{S}^2 = \mathrm{SO}(3)/\mathrm{SO}(2)$$

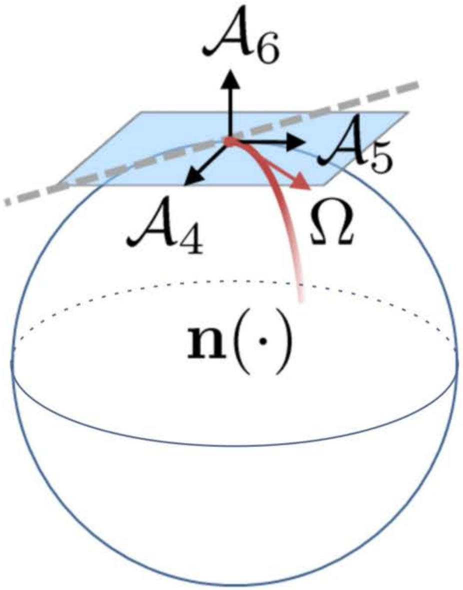


$\mathbf{x}'$ spatial velocity:

$\mathbf{x}' = \langle \omega^3|_\gamma, \gamma' \rangle\, \mathcal{A}_3|_\gamma = \mathcal{A}_3|_\gamma$

$\mathbf{x}''$ spatial curvature:

$\mathbf{x}'' = \langle \omega^5|_\gamma, \gamma' \rangle\, \mathcal{A}_1|_\gamma - \langle \omega^4|_\gamma, \gamma' \rangle\, \mathcal{A}_2|_\gamma$



Ω angular velocity:

$\Omega = \langle \omega^4, \gamma' \rangle \mathcal{A}_4 + \langle \omega^5, \gamma' \rangle \mathcal{A}_5$

PMEC(SE(3)): Sub-Riemannian problem in SE(3)

SR structure (SR manifold):

$$(M, \Delta, \mathcal{G}_\xi) \qquad M = \mathrm{SE}(3), \qquad \Delta = \mathrm{span}\{\mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\},$$
$$\mathcal{G}_\xi = \xi^2 \omega^3 \otimes \omega^3 + \omega^4 \otimes \omega^4 + \omega^5 \otimes \omega^5$$

SR distance (Carnot-Carathéodory distance):

$$d(g,h) = \min_{\substack{\gamma \in \mathrm{Lip}([0,T], \mathrm{SE}(3)), T \geq 0, \\ \dot{\gamma} \in \Delta,\ \gamma(0) = g, \gamma(T) = h}} \int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} \, \mathrm{d}t.$$

$\mathbf{P_{MEC}}$(SE(3)): to find a Lipschitzian curve $\gamma : [0,T] \to \mathrm{SE}(3)$, s.t.

$$\gamma(0) = e := (\mathbf{0}, I), \quad \gamma(T) = (\mathbf{x}_1, R_1) \in \mathrm{SE}(3),$$

$$\dot{\gamma}(t) \in \Delta \text{ for a.e. } t \in [0,T],$$

and $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)}(\dot{\gamma}(t), \dot{\gamma}(t))}\, \mathrm{d}t \to \min$ (with free T).

Optimal Control Formulation of SR-Problem in SE(3)

Control system

$$\dot{\gamma}(t) = u^3(t)\mathcal{A}_3|_{\gamma(t)} + u^4(t)\mathcal{A}_4|_{\gamma(t)} + u^5(t)\mathcal{A}_5|_{\gamma(t)}$$

Boundary conditions

$$\gamma(0) = e, \quad \gamma(T) = g_1 \in \mathrm{SE}(3)$$

Minimizing functional (here action functional)

$$\int_0^T \frac{1}{2}\left(\xi^2(u^3(t))^2 + (u^4(t))^2 + (u^5(t))^2\right)\, \mathrm{d}t \to \min .$$

In coordinates

$$\begin{cases} \dot{x} = u^3 \sin\beta, \\ \dot{y} = -u^3 \cos\beta \sin\theta, \\ \dot{z} = u^3 \cos\beta \cos\theta, \\ \dot{\theta} = \sec\beta(u^4 \cos\alpha - u^5 \sin\alpha), \\ \dot{\beta} = u^4 \sin\alpha + u^5 \cos\alpha, \\ \dot{\alpha} = -(u^4 \cos\alpha - u^5 \sin\alpha)\tan\beta, \end{cases}$$

$$(x(0), y(0), z(0), \theta(0), \beta(0), \alpha(0)) = \mathbf{0}$$

$$(x(T), y(T), z(T), \theta(T), \beta(T), \alpha(T)) = (x^1, y^1, z^1, \theta^1, \beta^1, \alpha^1)$$

- Complete controllability (Chow-Rashevski)
- Existance of minimizers (Filippov)
- No abnormal extremals: $\dim\,[\Delta, \Delta] = \dim\,(\mathrm{SE}(3))$
- The minimizers are analytic

Pontryagin Maximum Principle

- Left Invariant Hamiltonians $\lambda_i = \langle p, \mathcal{A}_i \rangle$, $i = 1, \ldots, 6$, where $p = p_1 \mathrm{d}x|_g + p_2 \mathrm{d}y|_g + p_3 \mathrm{d}z|_g + p_4 \mathrm{d}\theta|_g + p_5 \mathrm{d}\beta|_g + p_6 \mathrm{d}\alpha|_g$
- Control dependent Hamiltonian $H_u = u^3 \lambda_3 + u^4 \lambda_4 + u^5 \lambda_5 - \frac{1}{2}\left(\xi^2 (u^3)^2 + (u^4)^2 + (u^5)^2\right)$
- Maximality Condition $u^3 = \frac{\lambda_3}{\xi^2}, \quad u^4 = \lambda_4, \quad u^5 = \lambda_5.$
- The (maximized) Hamiltonian $H = \frac{1}{2}\left(\xi^{-2}\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- The Hamiltonian system of PMP (via Poisson brackets $\dot{\lambda}_i = \{H, \lambda_i\}$)

$$\begin{cases} \dot{\lambda}_1 = -\lambda_3 \lambda_5, \\ \dot{\lambda}_2 = \lambda_3 \lambda_4, \\ \dot{\lambda}_3 = \lambda_1 \lambda_5 - \lambda_2 \lambda_4, \\ \dot{\lambda}_4 = \frac{\lambda_2 \lambda_3}{\xi^2} - \lambda_5 \lambda_6, \\ \dot{\lambda}_5 = \lambda_4 \lambda_6 - \frac{\lambda_1 \lambda_3}{\xi^2}, \\ \dot{\lambda}_6 = 0, \end{cases}$$

— vertical part,

$$\begin{cases} \dot{x} = \frac{\lambda_3}{\xi^2} \sin \beta, \\ \dot{y} = -\frac{\lambda_3}{\xi^2} \cos \beta \sin \theta, \\ \dot{z} = \frac{\lambda_3}{\xi^2} \cos \beta \cos \theta, \\ \dot{\theta} = \sec \beta (\lambda_4 \cos \alpha - \lambda_5 \sin \alpha), \\ \dot{\beta} = \lambda_4 \sin \alpha + \lambda_5 \cos \alpha, \\ \dot{\alpha} = -(\lambda_4 \cos \alpha - \lambda_5 \sin \alpha) \tan \beta, \end{cases}$$

— horizontal part.

Liouville Integrability of the Hamiltonian System

First Integrals:

- the Hamiltonian $H = \frac{1}{2}\left(\lambda_3^2 + \lambda_4^2 + \lambda_5^2\right)$
- Left-invariant basis Hamiltonian λ_6
- Casimir functions $W = -\lambda_1\lambda_4 - \lambda_2\lambda_5 - \lambda_3\lambda_6, \quad \mathfrak{c}^2 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2$
- Right-invariant Hamiltonians

$$
\begin{aligned}
\rho_1 &= -\lambda_1 \cos\alpha \cos\beta + \lambda_2 \cos\beta \sin\alpha - \lambda_3 \sin\beta, \\
\rho_2 &= -\cos\gamma(\lambda_2 \cos\alpha + \lambda_1 \sin\alpha) + (\lambda_3 \cos\beta + (-\lambda_1 \cos\alpha + \lambda_2 \sin\alpha)\sin\beta)\sin\gamma, \\
\rho_3 &= -\lambda_3 \cos\beta \cos\gamma + \cos\gamma(\lambda_1 \cos\alpha - \lambda_2 \sin\alpha)\sin\beta - (\lambda_2 \cos\alpha + \lambda_1 \sin\alpha)\sin\gamma, \\
&\rho_4, \quad \rho_5, \quad \rho_6.
\end{aligned}
$$

Complete system of first Integrals: $\mathrm{I} = (H,\, \lambda_6,\, W,\, \rho_1,\, \rho_2,\, \rho_3)$

$$
\{I_i, I_j\} = 0 \qquad \frac{\partial(\rho_1,\rho_2,\rho_3,W,H,\lambda_6)}{\partial(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6)}(q,\lambda) = \quad -\lambda_2\lambda_4 + \lambda_1\lambda_5 \neq 0
$$

Theorem *The Hamiltonian system of PMP for sub-Riemanian problem on* SE(3) *is Liouville integrable.*

Integration of the Vertical Part

Theorem *Suppose $\lambda_6(0) = 0$; then the vertical part is given by*

$$\dot{\lambda}_1 = -\lambda_3\lambda_5, \quad \dot{\lambda}_2 = \lambda_3\lambda_4, \quad \dot{\lambda}_3 = \lambda_1\lambda_5 - \lambda_2\lambda_4, \quad \dot{\lambda}_4 = \lambda_2\lambda_3, \quad \dot{\lambda}_5 = -\lambda_1\lambda_3.$$

The momenta λ_4, λ_5 are expressed via $U(t) = \int_0^t \lambda_3(\tau)\,\mathrm{d}\,\tau$ and the initial values

$$\lambda_4(t) = \frac{\lambda_2(0) + \lambda_4(0)}{2} \exp\left(U\left(t\right)\right) - \frac{\lambda_2(0) - \lambda_4(0)}{2} \exp\left(-U\left(t\right)\right),$$
$$\lambda_5(t) = \frac{\lambda_1(0) + \lambda_5(0)}{2} \exp\left(-U\left(t\right)\right) - \frac{\lambda_1(0) - \lambda_5(0)}{2} \exp\left(U\left(t\right)\right).$$

The momentum λ_3 is expressed via the initial values depending on several cases. For the cases $\lambda_1(0) = \pm\lambda_5(0)$, $\lambda_2(0) = \mp\lambda_4(0)$, we have

$$\lambda_3(t) = \frac{(b + \lambda_3(0))\,\mathrm{e}^{\pm bt} - (b - \lambda_3(0))\,\mathrm{e}^{\mp bt}}{\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}}, \qquad U(t) = -\ln\left(\frac{1}{2}\left[\left(1 + \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\pm bt} + \left(1 - \frac{\lambda_3(0)}{b}\right)\mathrm{e}^{\mp bt}\right]\right),$$

where $b = \sqrt{\lambda_3^2(0) + \lambda_4^2(0) + \lambda_5^2(0)}$. Otherwise, we have

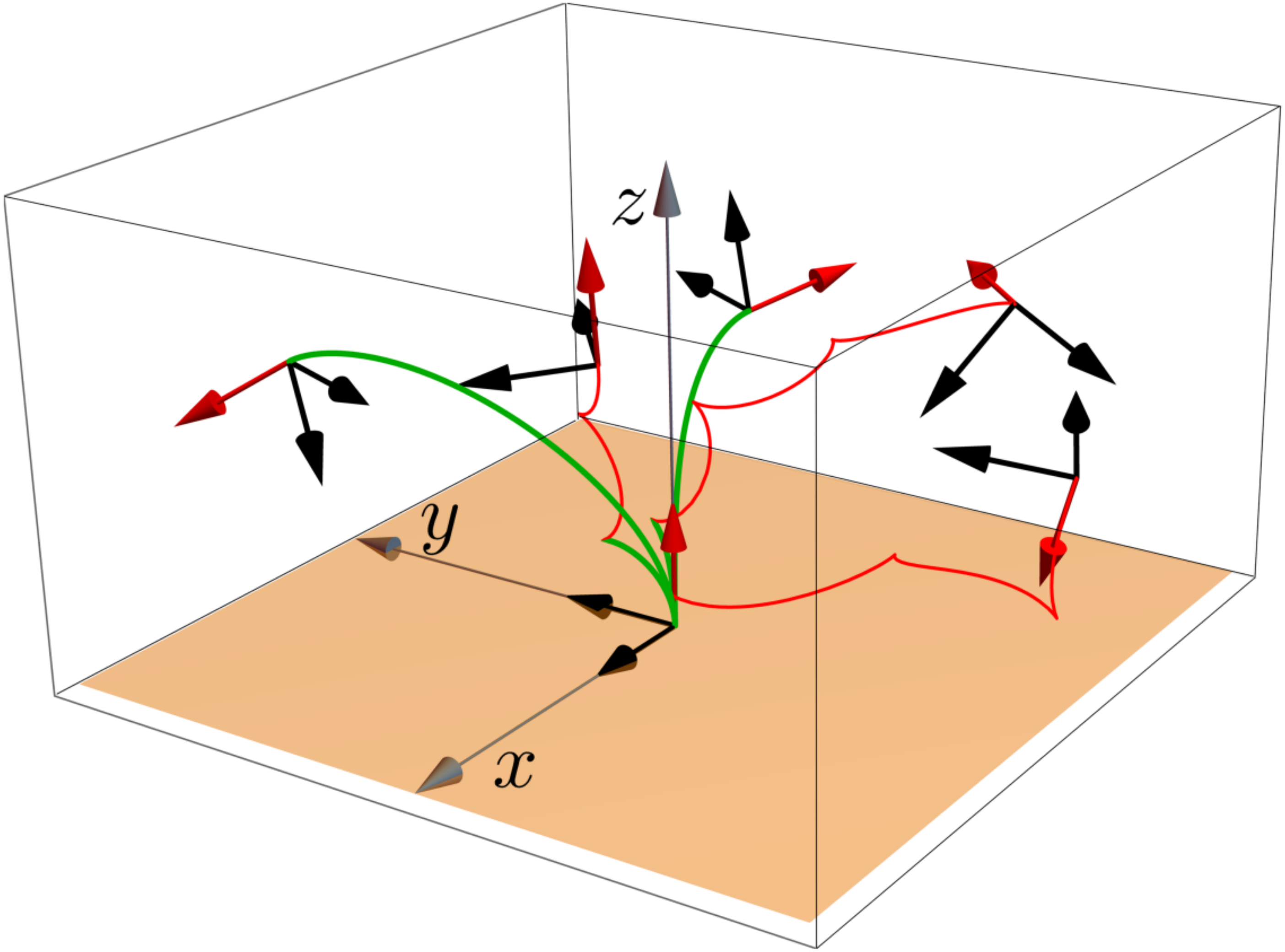
$$\lambda_3(t) = -\frac{P}{2}\,\mathrm{sn}\left(\psi_t, k\right), \qquad U(t) = \frac{1}{2}\ln\left(\frac{A}{B} + \frac{P^2}{2B}\left(\mathrm{cn}^2\left(\psi_t, k\right) + \frac{1}{k}\,\mathrm{cn}\left(\psi_t, k\right)\mathrm{dn}\left(\psi_t, k\right)\right)\right),$$

where $A = (\lambda_1(0) + \lambda_5(0))^2 + (\lambda_2(0) - \lambda_4(0))^2$, $B = (\lambda_1(0) - \lambda_5(0))^2 + (\lambda_2(0) + \lambda_4(0))^2$,
$P = \sqrt{4\lambda_3^2\left(0\right) + \left(\sqrt{A} - \sqrt{B}\right)^2}$, $Q = \sqrt{4\lambda_3^2\left(0\right) + \left(\sqrt{A} + \sqrt{B}\right)^2}$,

$$\psi_t = F\left(p_0, k\right) + \tfrac{Q}{2}t, \quad k = \tfrac{P}{Q}, \qquad p_0 = \begin{cases} -\arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B \geq A, \\ \pi + \arcsin\left(\frac{2\lambda_3(0)}{P}\right), & \text{if } B < A. \end{cases}$$

Geodesics in **Pcurve($\mathbb{R}^3$)** as Projection of SR geodesics in SE(3)

Spatial projection of SR geodesics in SE(3) can have singularities (the cusp points)



SR problem Pmec($\mathbf{R}^3$ x $\mathbf{S}^2$) in Quotient SE(3)/({0}xSO(2))

Well-defined distance on the quotient $\mathbb{R}^3 \rtimes S^2$

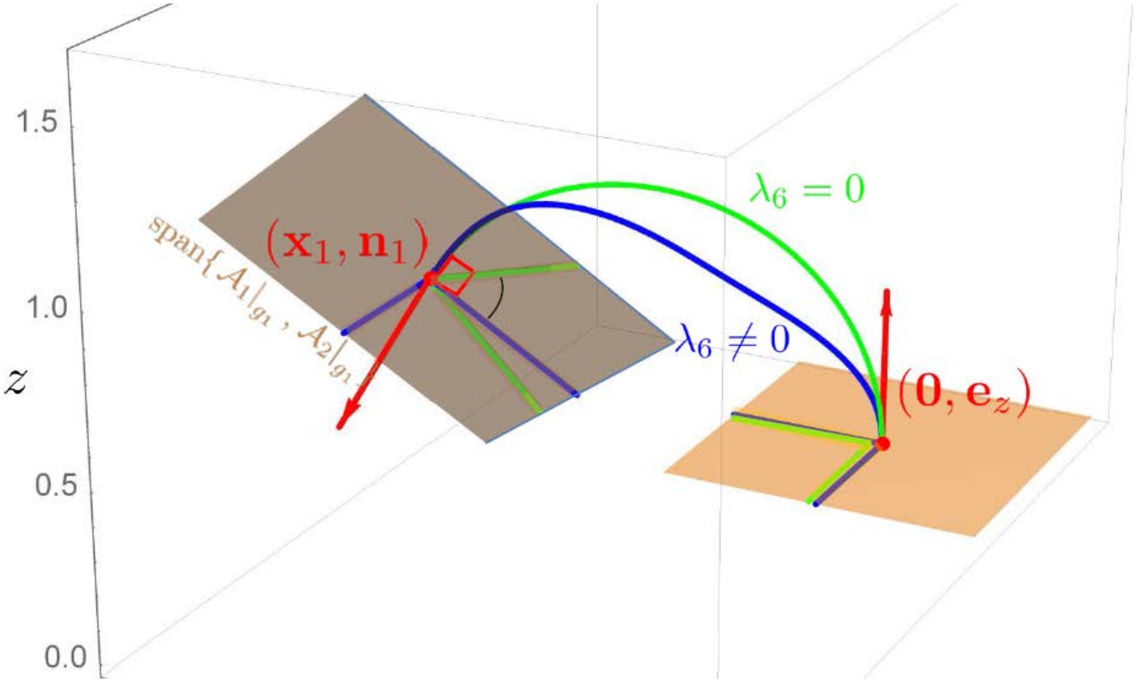
$$\begin{aligned} d_{\mathbb{R}^3 \rtimes S^2}((\mathbf{0}, \mathbf{e}_z), (\mathbf{y}_1, \mathbf{n}_1)) &= \min_{\alpha^1, \alpha^2 \in [0,2\pi)} d(e h_{\alpha^1}, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2}) \\ &= \min_{\alpha^1, \alpha^2 \in [0,2\pi)} d(e, h_{\alpha^1}^{-1} (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha^2 - \alpha^1} h_{\alpha^1}) \\ &= \min_{\alpha \in [0,2\pi)} d(e, (\mathbf{y}_1, R_{\mathbf{n}_1}) h_{\alpha}) \end{aligned}$$

$\boxed{\mathbf{P_{mec}}(\mathbb{R}^3 \times S^2):}$ Let $(\mathbf{y}_1, \mathbf{n}_1) \in \mathbb{R}^3 \rtimes S^2$. Find

$$[0, T] \ni t \mapsto (\mathbf{x}(t), \mathbf{n}(t)) = \gamma(t) \odot (\mathbf{0}, \mathbf{e}_z) \in \mathbb{R}^3 \rtimes S^2,$$

with γ a Lipschitzian curve in SE(3) with velocity $\dot{\gamma} \in \Delta$, such that sub-Riemannian length $\int_0^T \sqrt{\mathcal{G}_\xi|_{\gamma(t)} (\dot{\gamma}(t), \dot{\gamma}(t))} \mathrm{d}t$ is minimal under boundary conditions $\gamma(0) = (\mathbf{0}, I)$ and $\gamma(T) = (\mathbf{y}_1, R_{\mathbf{n}_1} R_{\mathbf{e}_z, \alpha})$, where both $T \geq 0$ and $\alpha \in [0, 2\pi)$ are free variables in the optimization process.

Relation of Pcurve($\mathbf{R}^3$), Pmec($\mathbf{R}^3 \times \mathbf{S}^2$) and PMEC(SE(3))



Theorem *If* $g_1 = (\mathbf{x}_1, R_1) \in \mathrm{SE}(3)$ *is chosen s.t. a corresponding minimizer* γ^* *of* $\mathbf{P}_{\mathbf{MEC}}$ *satisfies* $u^3(t) := \langle \omega^3|_{\gamma^*(t)}, \dot{\gamma}^*(t)\rangle > 0,\ t \in (0,T)$*, then* γ^* *can be parameterized by spatial arclength* s*, and its spatial projection does not exhibit a cusp. If moreover* g_1 *is chosen s.t.* γ^* *has* $\lambda_6(0) = 0$ *then this yields the required minimum choice of* α*, and* $\gamma^*(t)$ *provides the minimizer* $(\mathbf{x}^*(t), \mathbf{n}^*(t)) = \gamma^*(t) \odot (\mathbf{0}, \mathbf{e}_z)$ *of* $\mathbf{P}_{\mathbf{mec}}$.

Under these two requirements the spatial projection $\mathbf{x}^*(\cdot)$ *of* $\gamma^*(\cdot) = (\mathbf{x}^*(\cdot), R^*(\cdot))$ *coincides with a minimizer of problem* $\mathbf{P}_{\mathbf{curve}}$.

Explicit Expression for Geodesics

Theorem *The spatial part of the cuspless sub-Riemannian geodesics in* $\mathbf{P}_{\mathbf{mec}}$ *is given by*

$$\boxed{\boldsymbol{x}(s) = \tilde{R}(0)^T(\tilde{\boldsymbol{x}}(s) - \tilde{\boldsymbol{x}}(0)),}$$

where $\tilde{R}(0)$ *and* $\tilde{\boldsymbol{x}}(s) := (\tilde{x}(s), \tilde{y}(s), \tilde{z}(s))$ *are given in terms of* $\underline{\lambda}^{(1)}(0)$ *and* $\underline{\lambda}^{(2)}(0)$ *depending on several cases.*
For all cases with $\underline{\lambda}^{(1)}(0) \neq \underline{\lambda}^{(2)}(0)$ *we have* $\boxed{\tilde{x}(s) = \frac{1}{\mathfrak{c}}\int_0^s \lambda_3(\tau)\,\mathrm{d}\tau = -\frac{i\sqrt{1-d}\sqrt{1+\mathfrak{c}^2}}{\mathfrak{c}\sqrt{2}}\left(E\left(\left(s+\frac{\varphi}{2}\right)i, M\right) - E\left(\frac{\varphi}{2}i, M\right)\right)}$,
where $M := \frac{2d}{d-1}$, $d := \frac{\|\underline{\lambda}^{(2)}(0)+\underline{\lambda}^{(1)}(0)\|\|\underline{\lambda}^{(2)}(0)-\underline{\lambda}^{(1)}(0)\|}{1+\mathfrak{c}^2} \leq 1$, *and* $\varphi := \log\frac{\|\underline{\lambda}^{(2)}(0)+\underline{\lambda}^{(1)}(0)\|}{\|\underline{\lambda}^{(2)}(0)-\underline{\lambda}^{(1)}(0)\|}$.

For the case $\underline{\lambda}^{(1)}(0) = \mathbf{0}$, *we have* $\tilde{R}(0) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \in \mathrm{SO}(3)$, $\quad \begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{-1}{\mathfrak{c}}\begin{pmatrix} \lambda_4(s) \\ \lambda_5(s) \end{pmatrix}$.

For the case $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$, *we have* $\tilde{R}(0) = \frac{1}{\mathfrak{c}}\begin{pmatrix} \lambda_1(0) & \lambda_2(0) & \lambda_3(0) \\ \mathfrak{c}\frac{-\lambda_2(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \mathfrak{c}\frac{\lambda_1(0)}{\|\underline{\lambda}^{(1)}(0)\|} & 0 \\ \frac{-\lambda_1(0)\lambda_3(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \frac{-\lambda_2(0)\lambda_3(0)}{\|\underline{\lambda}^{(1)}(0)\|} & \|\underline{\lambda}^{(1)}(0)\| \end{pmatrix} \in \mathrm{SO}(3)$.

For the case $W = 0$ *along with* $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$, *we have* $\begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{\underline{\lambda}^{(2)}(s)\cdot\underline{\lambda}^{(1)}(0)}{\mathfrak{c}\|\underline{\lambda}^{(1)}(0)\|}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

For $W \neq 0$ *along with* $\underline{\lambda}^{(1)}(0) \neq \mathbf{0}$ *we have*

$$\boxed{\begin{pmatrix} \tilde{y}(s) \\ \tilde{z}(s) \end{pmatrix} = \frac{\sqrt{\|\underline{\lambda}^{(2)}(s)\|^2 - W^2\mathfrak{c}^{-2}}}{\mathfrak{c}^2\|\underline{\lambda}^{(1)}(0)\|\sqrt{\|\underline{\lambda}^{(2)}(0)\|^2 - W^2\mathfrak{c}^{-2}}}\begin{pmatrix} \cos\tilde{\psi}(s) & -\sin\tilde{\psi}(s) \\ \sin\tilde{\psi}(s) & \cos\tilde{\psi}(s) \end{pmatrix}\begin{pmatrix} W\lambda_3(0) \\ \mathfrak{c}(\underline{\lambda}^{(2)}(0)\cdot\underline{\lambda}^{(1)}(0)) \end{pmatrix}}, \textit{ where}$$

$$\tilde{\psi}(s) = \int_0^s \frac{W\mathfrak{c}^{-1}\lambda_3(\tau)}{\|\underline{\lambda}^{(2)}(\tau)\|^2 - W^2\mathfrak{c}^{-2}}\,\mathrm{d}\tau = -\frac{W}{\mathfrak{c}}\frac{\sqrt{2}}{\sqrt{1+\mathfrak{c}^2}\sqrt{1-d}}\frac{1}{i}(F(i(s+\frac{\varphi}{2}), M) - F(\frac{i\varphi}{2}, M)$$
$$-(1-\frac{1}{D})(\Pi\left(\frac{M}{D}, i(s+\frac{\varphi}{2}), M\right) - \Pi\left(\frac{M}{D}, \frac{i\varphi}{2}, M\right))),$$

with $D = 2(\frac{W^2}{\mathfrak{c}^2} - 1)(1+\mathfrak{c}^2)^{-1}(1-d)^{-1} + 1$ *and* $|\tilde{\psi}(s)| < \pi$, $\mathrm{sign}(\tilde{\psi}(s)) = \mathrm{sign}(W)$.

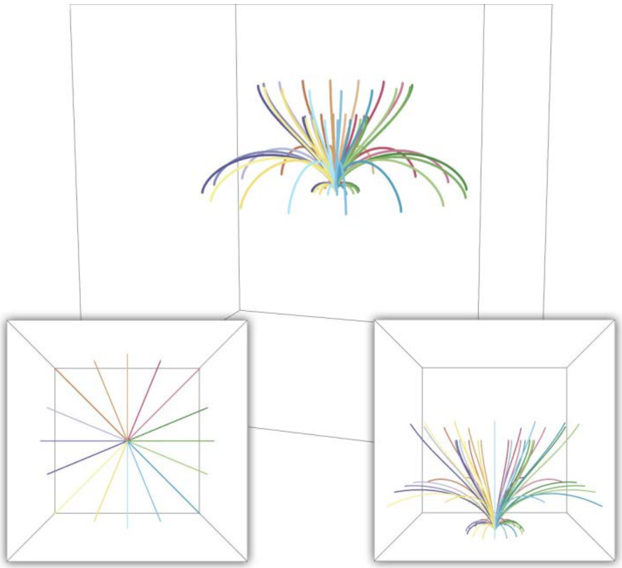
Geometric Properties of Geodesics

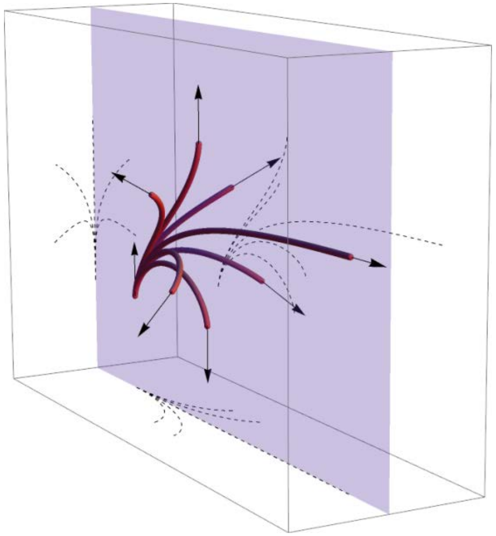
Corollary 1 *The absolute curvature and the signed torsion of a geodesic of* $\mathbf{P}_{\mathbf{curve}}$ *are given by* $\kappa = \frac{\sqrt{\lambda_4^2+\lambda_5^2}}{\lambda_3} = \frac{\sqrt{1-\lambda_3^2}}{\lambda_3}$, $\tau = \frac{W}{\lambda_4^2+\lambda_5^2}$. *Thus, the torsion is bounded as* $|W| \leq |\tau(s)| \leq \frac{2|W|}{\sqrt{(1-\mathfrak{c}^2)^2+4W^2}+1-\mathfrak{c}^2}$ *for all* $0 \leq s \leq s_{max}$.

Corollary 2 *The cuspless spatial projections of sub-Riemannian geodesics of* $\mathbf{P}_{\mathbf{mec}}$ *(i.e. geodesics of* $\mathbf{P}_{\mathbf{curve}}$*) are planar if and only if* $W = 0$.

Corollary 3 *Given admissible coplanar end conditions for* $\mathbf{P}_{\mathbf{curve}}$, *the unique cuspless geodesic connecting them is planar.*

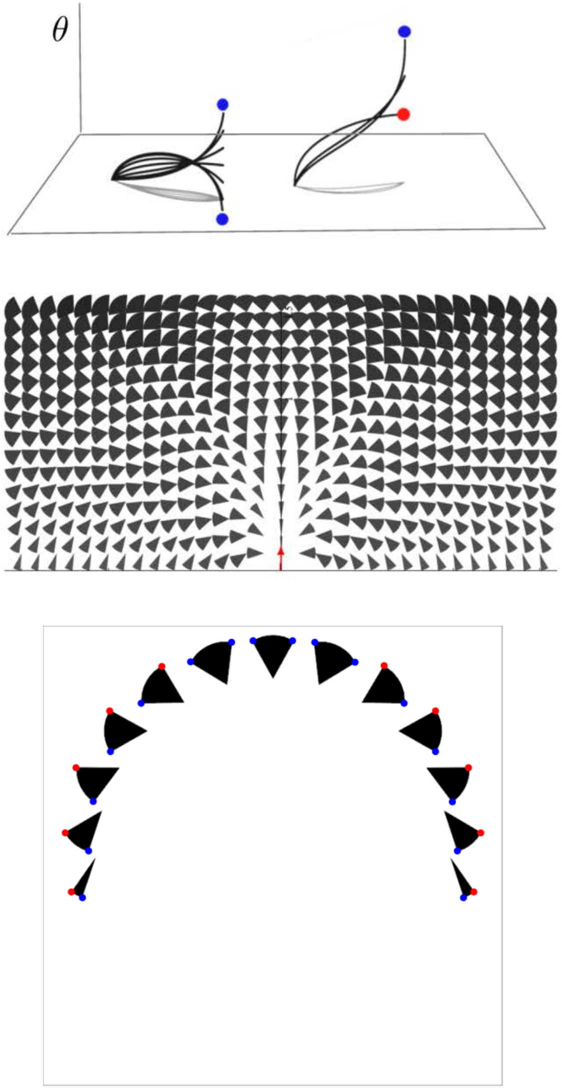
Corollary 4 *All cuspless sub-Riemannian geodesics in* $(\mathrm{SE}(3), \Delta, \mathcal{G}_1)$ *with* $\lambda_6 = 0$ *and* $\sum_{i=1}^{3} \lambda_i^2(0) \neq 0$, *departing from* $e = (\mathbf{0}, I)$ *stay in the upper half space* $z \geq 0$.



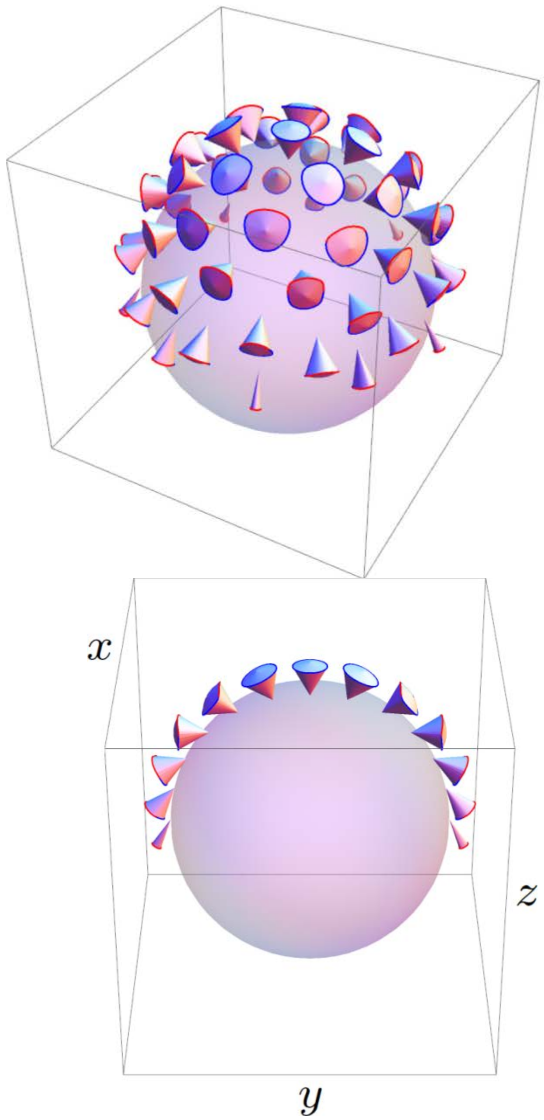


Range of Exponential Map of Pcurve

$\mathbb{R}^2 \rtimes S^1 = SE(2)$



$\mathbb{R}^3 \rtimes S^2 = SE(3)/(\{0\} \times SO(2))$



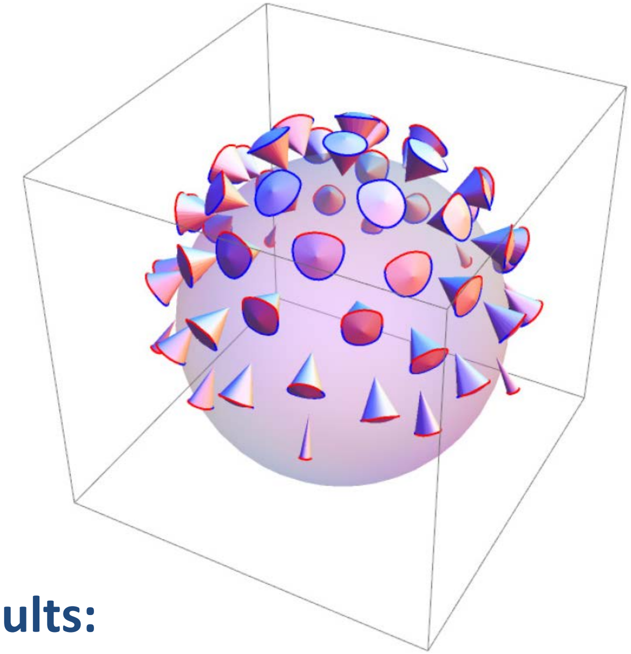
Let $\mathcal{R}$ denotes the range, and $\mathcal{D}_0$ the domain of exponential map $\mathbf{P}_{\mathbf{curve}}$.

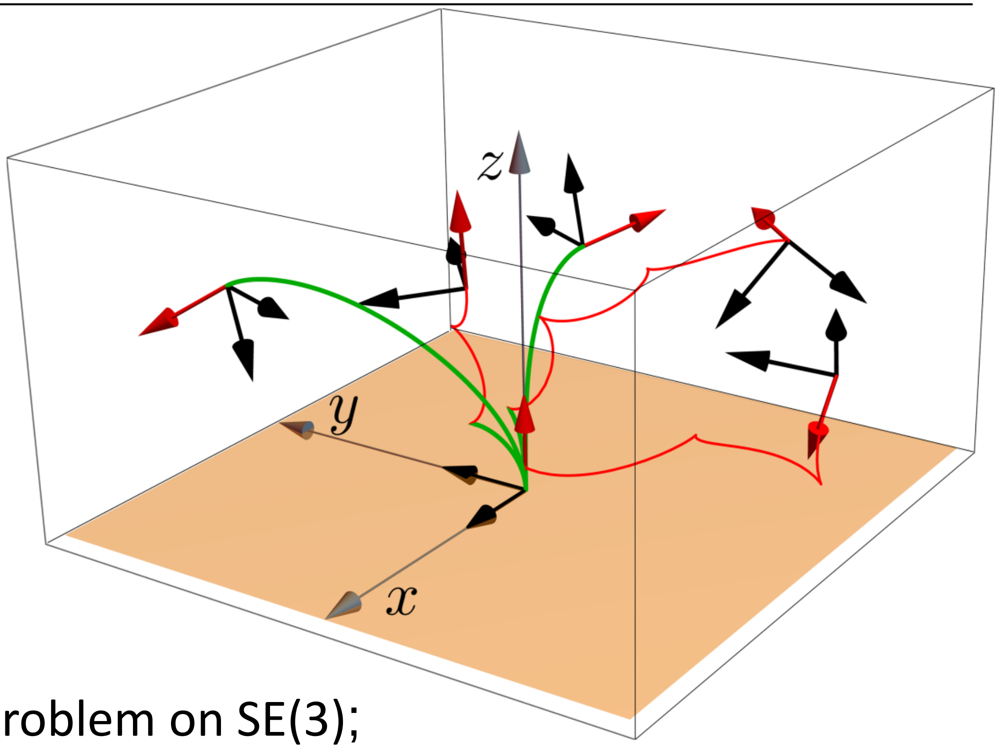
Conjecture:

$Exp : \mathcal{D}_0 \to \mathcal{R}$ is a homeomorphism

$Exp : int(\mathcal{D}_0) \to int(\mathcal{R})$ is a diffeomorphism

SR-geodesics on SE(3) with cuspless spatial projections





Results:

- Lift $P_{\mathrm{curve}}(\mathbb{R}^3)$ to sub-Riemannian problem on SE(3);
- Hamiltonian system of PMP;
- Liouville integrability of the Hamiltonian system;
- Explicit expressions for SR-geodesics in spatial arclength parameterization;
- Evaluation of first cusp time;
- Admissible boundary conditions reachable by cuspless geodesics;
- Geometrical properties: bounds on torsion, planarity conditions, symmetries;
- Numerical investigation of absence of conjugate points;

Thank you for your attention!

Partial Cartan Connection

Partial Cartan connection $\overline{\nabla}$ on the tangent bundle of $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$

$$\overline{\nabla}_{\dot{\gamma}}\mathcal{A} := \sum_{k=3}^{5} \left((\dot{a}^k) - \sum_{i,j=3}^{5} c^k_{i,j} \, (\dot{\gamma}^i) a^j \right) \mathcal{A}_k,$$

with $\dot{\gamma} = \sum\limits_{i=3}^{5} \dot{\gamma}^i \mathcal{A}_i|_\gamma$, $\quad \mathcal{A} = \sum\limits_{k=3}^{5} a^k \mathcal{A}_k$,

and Lie algebra structure constants $c^k_{i,j}$.

Horizontal exponential curves $t \mapsto g_0 \, e^{t \sum_{i=3}^{5} c^i A_i}$ with $\xi^2 (c^3)^2 + (c^4)^2 + (c^5)^2 = 1$, $g_0 \in \mathrm{SE}(3)$ are the auto-parallel curves, i.e. $\boxed{\overline{\nabla}_{\dot{\gamma}} \dot{\gamma} = 0}$

Partial Cartan connection $\overline{\nabla}^*$ on the cotangent bundle of $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$

$$\overline{\nabla}^*_{\dot{\gamma}}\lambda := \sum_{i=1}^{6} \left(\dot{\lambda}_i + \sum_{j=3}^{5} \sum_{k=1}^{6} c^k_{i,j} \lambda_k \, \dot{\gamma}^j \right) \omega^i$$

with $\dot{\gamma} = \sum\limits_{i=3}^{5} \dot{\gamma}^i \mathcal{A}_i|_\gamma$, $\quad \lambda = \sum_{i=1}^{6} \lambda_i \omega^i|_\gamma$,

and Lie algebra structure constants $c^k_{i,j}$.

Along SR geodesics one has covariantly constant momentum $\boxed{\overline{\nabla}^*_{\dot{\gamma}}\lambda = 0}$ and $\boxed{\mathcal{G}_\xi^{-1}\left(\sum\limits_{i=3}^{5} \lambda_i \omega^i \right) = \dot{\gamma}}$

Group representation $m : \mathrm{SE}(3) \to \mathrm{Aut}(\mathbb{R}^6)$ visible in the Cartan-matrix

$$m(\mathbf{x}, R) := \begin{pmatrix} R & \sigma_{\mathbf{x}} R \\ 0 & R \end{pmatrix}, \text{ with } \mathbf{x} = \sum_{i=1}^{3} x^i \mathbf{e}_i, \; \sigma_{\mathbf{x}} = \sum_{i=1}^{3} x^i A_{3+i} \in \mathrm{so}(3), \text{ s.t. } \sigma_{\mathbf{x}} \mathbf{y} = \mathbf{x} \times \mathbf{y}$$

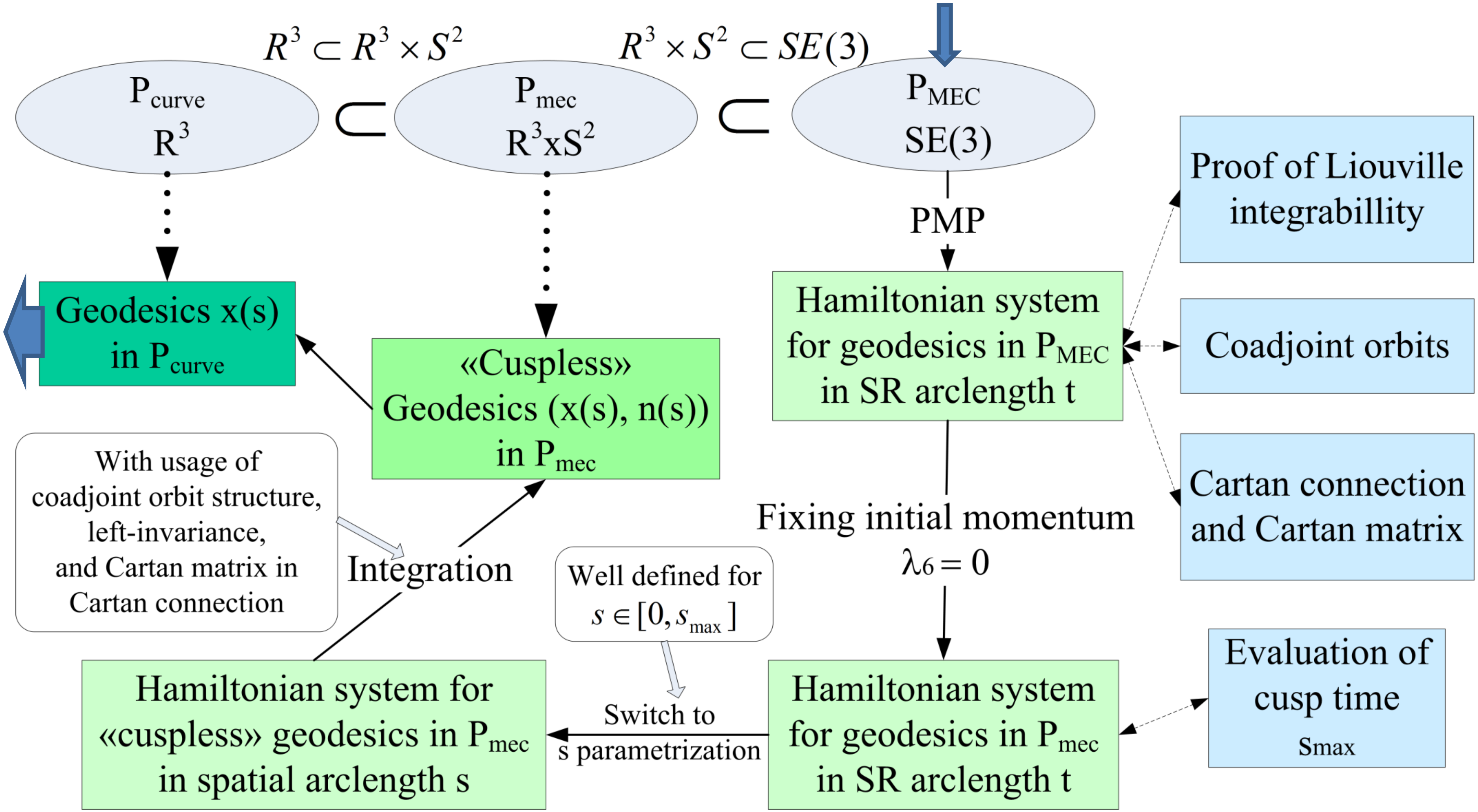
Theorem *Let m be our matrix group representation, s.t.*

$$\mathrm{d}\boldsymbol{\lambda}|_\gamma = \boldsymbol{\lambda}|_\gamma \; m(\gamma^{-1}) \mathrm{d}m(\gamma).$$

Then along the SR geodesics in $(\mathrm{SE}(3), \Delta, \mathcal{G}_\xi)$ the following relation holds:

$$\boldsymbol{\lambda}(t) m(\gamma(t))^{-1} = \boldsymbol{\lambda}(0) \, m(\gamma(0))^{-1} = \boldsymbol{\lambda}(0).$$

Problem **Pcurve(R^3)**, **Pmec(R^3xS^2)** and **PMEC(SE(3))**



$R^3 \subset R^3 \times S^2$

$R^3 \times S^2 \subset SE(3)$

P_{curve}
R^3

$\subset$

P_{mec}
R^3xS^2

$\subset$

P_{MEC}
SE(3)

PMP

Hamiltonian system
for geodesics in P_{MEC}
in SR arclength t

Proof of Liouville
integrabillity

Coadjoint orbits

Cartan connection
and Cartan matrix

Geodesics x(s)
in P_{curve}

«Cuspless»
Geodesics (x(s), n(s))
in P_{mec}

With usage of
coadjoint orbit structure,
left-invariance,
and Cartan matrix in
Cartan connection

Integration

Fixing initial momentum
$\lambda_6 = 0$

Well defined for
$s \in [0, s_{\max}]$

Switch to
s parametrization

Hamiltonian system for
«cuspless» geodesics in P_{mec}
in spatial arclength s

Hamiltonian system
for geodesics in P_{mec}
in SR arclength t

Evaluation of
cusp time
Smax

Analysis of Images of the Retina

Diabetic retinopathy --- one of the main causes of blindness.
Epidemic forms: 10% people in China suffer from DR.
Patients are found early --> treatment is well possible.
Early warning --- leakage and malformation of blood vessels.
The retina --- excellent view on the microvasculature of the brain.

Healthy retina

Diabetes Retinopathy with tortuous vessels