

# Geometry and Control

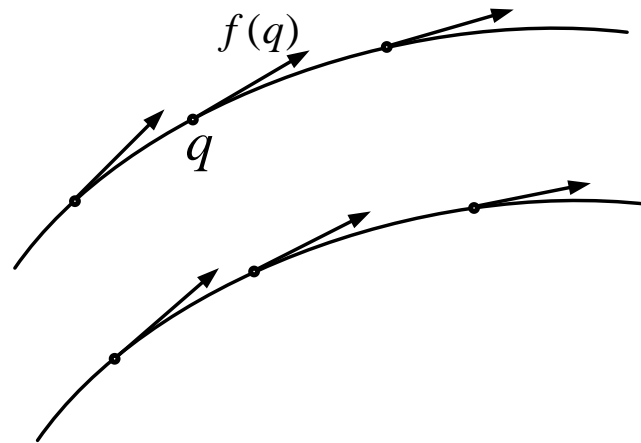
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<http://gct.math.nsc.ru>



Smooth dynamical system:

$$\dot{q}(t) = f(q(t)), \quad q \in M, \quad t \in \mathbb{R},$$



generates a flow

$$P^t : M \rightarrow M, \quad P^t : q(0) \mapsto q(t), \quad t \in \mathbb{R}.$$

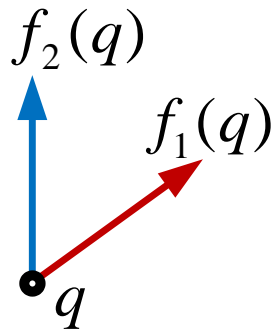
Control system:

$$\dot{q} = f_u(q), \quad u \in U.$$

Control:  $t \mapsto u(t)$ ,  $t \geq 0$ .

Trajectory:  $t \mapsto q(t)$ , where  $\dot{q}(t) = f_{u(t)}(q(t))$ .

Special case:  $U = \{1, 2\}$ :



Trajectories:

**Example.** Unicycle:

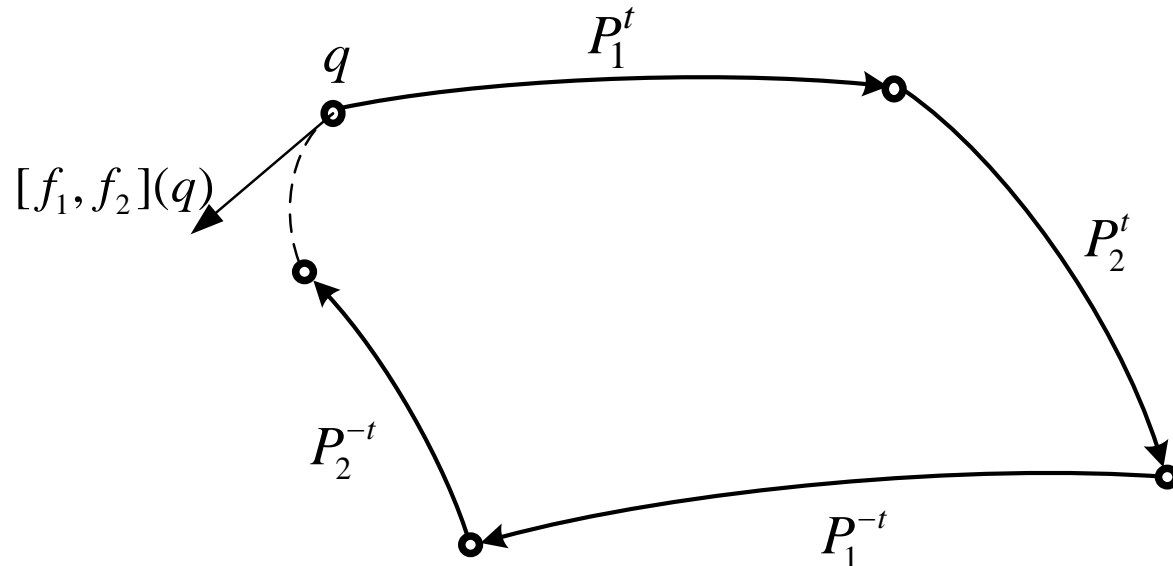
$$q = (x, \theta) : \begin{array}{c} \nearrow \\ \cdot \\ \text{---} \theta \\ \cdot \\ x \end{array}, M = \mathbb{R}^2 \times S^1.$$

$$P_1^t : \quad , \quad P_2^t :$$

Third directions is the “parallel parking”:

We realize it using only  $P_1^{\pm t}$  and  $P_2^{\pm t}$ :

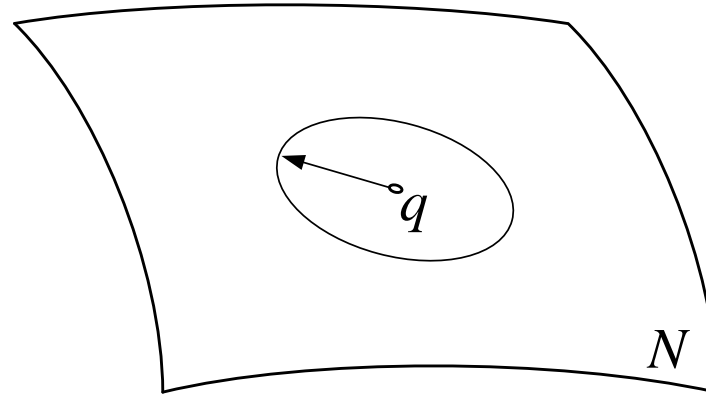
The commutator of flows and vector fields:



$$P_2^{-t} \circ P_1^{-t} \circ P_2^t \circ P_1^t(q) = q + t^2 [f_1, f_2](q) + O(t^3).$$

In the example, the field  $g = [f_1, f_2]$  generates the parallel parking.

Similar problem on a non flat surface:



$M$  is the “spherical bundle” of all tangent vectors of the length 1 to the surface  $N$ .

The flow  $P_1^t$  is as before,  $P_2^t$  is the geodesic flow:

$$P_2^t : (x(0), \dot{x}(0)) \mapsto (x(t), \dot{x}(t)),$$

where  $\ddot{x}(t) \perp N$ .

Then  $g = [f_1, f_2]$  generates the “parallel parking” .

The difference between the flat and non flat cases:

“translation”  $f_2$  and “parallel parking”  $g$  obviously commute in the flat case.

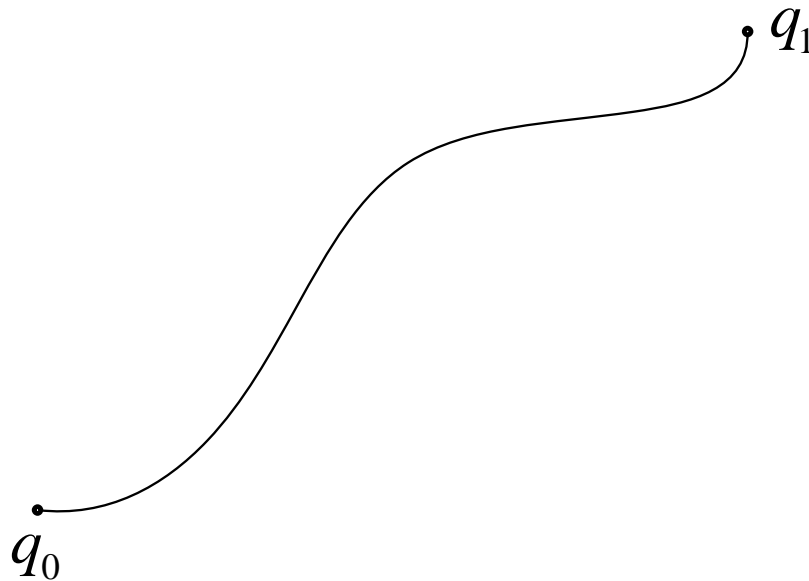
In general,  $[g, f_2] = \kappa f_1$ , where  $\kappa$  is the curvature!

In higher dimensions, the field  $[g, f_2]$  may be linearly independent on  $f_1, f_2, [f_1, f_2] = g$ .

General case:

$$\text{Lie}_q f_U = \text{span}\{[f_{u_1}, [\dots, f_{u_k}]\dots](q) : u_i \in U, k = 1, 2, \dots\}.$$

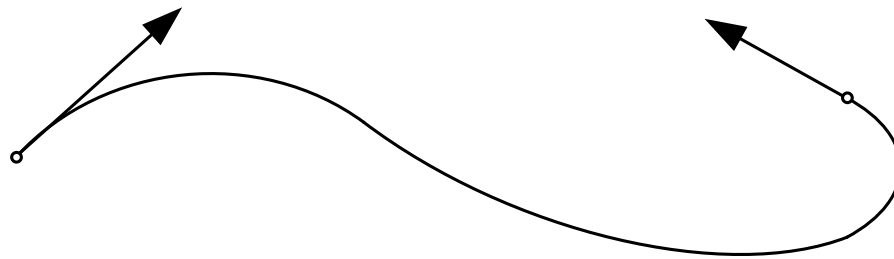
**Theorem** (Rashevskij–Chow). *If  $\text{Lie}_q f = T_q M$ ,  $\forall q \in M$ , then  $\forall q_0, q_1 \in M \exists t \mapsto (u(t), q(t))$  such that  $\dot{q}(t) = f_{u(t)}(q(t))$ ,  $q(0) = q_0$ ,  $q(t_1) = q_1$ .*



## Optimal control

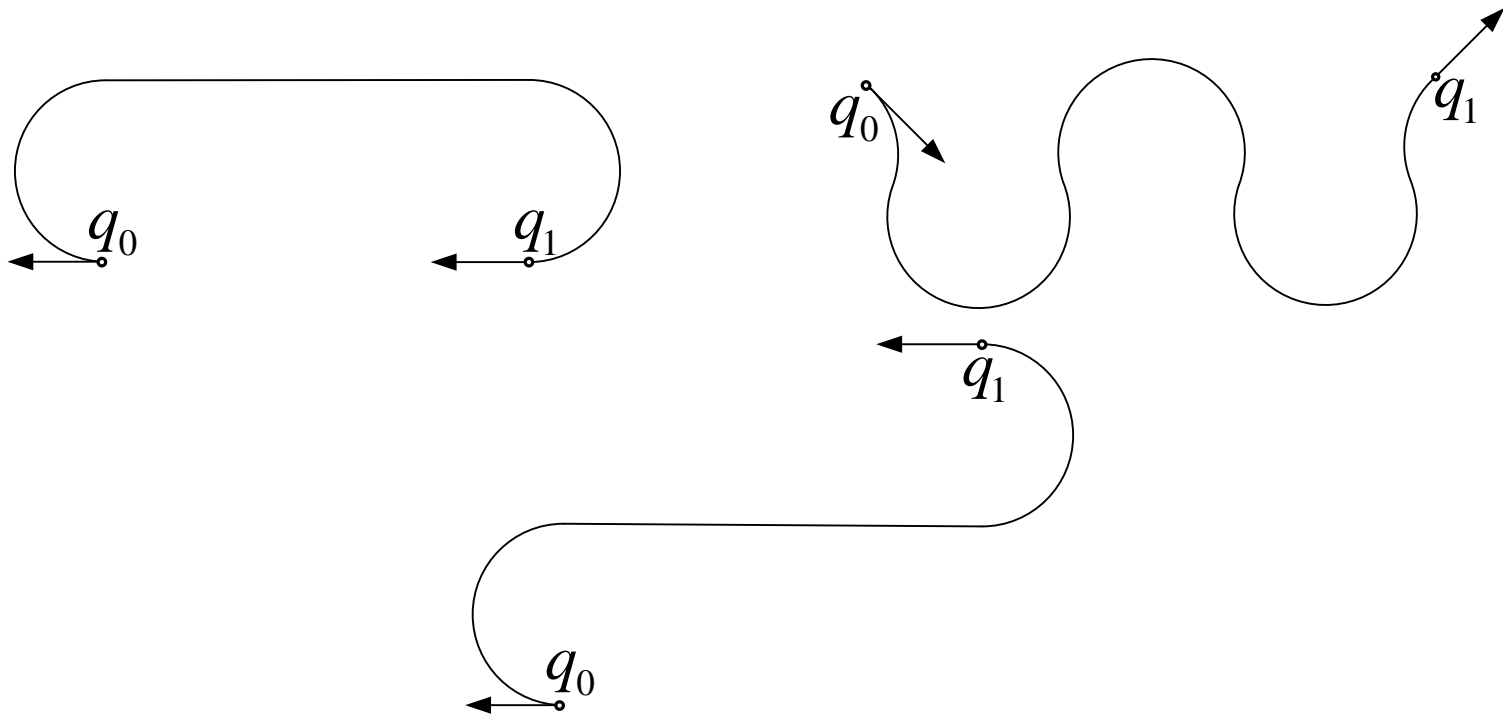
Example.  $M = \mathbb{R}^2 \times S^1$ ,  $\dot{q} = uf_1(q) + f_2(q)$ .

Admissible trajectories  $t \mapsto q(t) = (x(t), \theta(t))$ , where  $\dot{x} = \begin{pmatrix} \cos \theta \\ \sin \theta \end{pmatrix}$  and  $\dot{\theta} = u$  is the curvature of the curve  $x(\cdot)$ .



I) Markov–Dubins problem:  $|u| \leq c$ , minimise the length of  $x(\cdot)$  (= time  $t_1$ ).

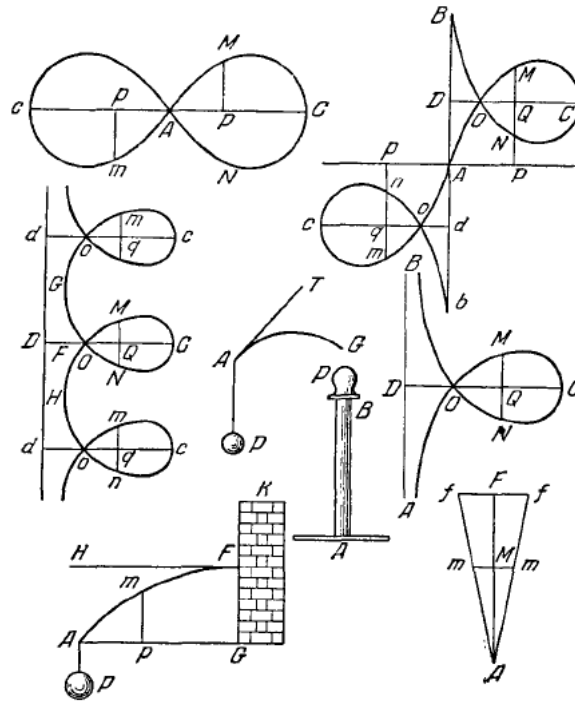
“Geodesics” are special concatenations of circle and linear segments:



Optimal piece has no more than 3 switchings.

II) Euler elastic problem:  $u \in \mathbb{R}$ , minimise  $\int_0^{t_1} u^2(t) dt$ .

“Geodesics” or *elasticae*: the curves whose curvature is a linear function of coordinates,  $u(t) = \langle a, x(t) \rangle + \alpha$ .



Both problems are translation and rotation invariant, where  $S^1 = \text{SO}(2)$  is the group of rotations.

**Example** (rolling without slipping or twisting).

Here  $M = \mathbb{R}^2 \times \text{SO}(3)$ ,  $q = (x, X)$ , where  $x \in \mathbb{R}^2$ ,  $x \in \text{SO}(3)$ .

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \dot{x} = u, \quad \dot{q} = u_1 F_1(q) + u_2 F_2(q).$$

Minimize  $\int_0^{t_1} |u(t)| dt$ , the length of  $x(\cdot)$ .

Minimal length:

$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)| dt : q(0) = q_0, q(t_1) = q_1 \right\}$$

is a metric on  $M$ :

$$\delta(q_0, q_1) = \delta(q_1, q_0), \quad \delta(q_0, q_2) \leq \delta(q_0, q_1) + \delta(q_1, q_2).$$

“Rolling geodesics” are again elasticas.

General sub-Riemannian problem:

$$\dot{q} = \sum_{i=1}^k u_i F_i(q), \quad q \in M, \quad u = (u_1, \dots, u_k)^T \in \mathbb{R}^k,$$

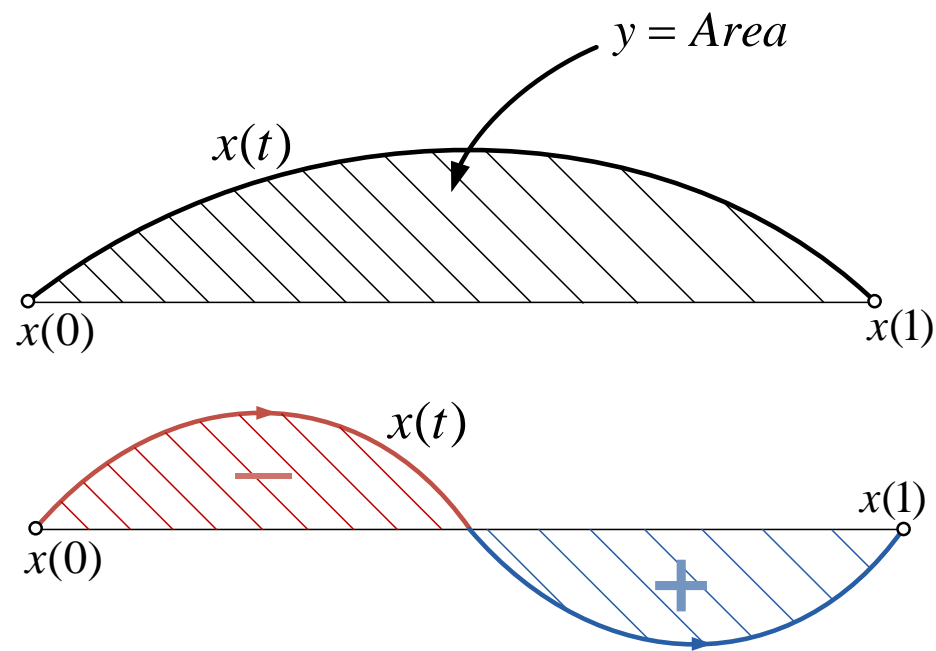
defines a “Carnot–Caratheodory metric” on  $M$ :

$$\delta(q_0, q_1) = \inf \left\{ \int_0^{t_1} |u(t)| dt : q(0) = q_0, q(t_1) = q_1 \right\}$$

### 3-dim examples:

I) Dido problem.  $q = (x, y)$ ,  $x \in \mathbb{R}^2$ ,  $y \in \mathbb{R}$ ,

$$\begin{cases} \dot{x} = u \\ 2\dot{y} = x \wedge u \end{cases}$$

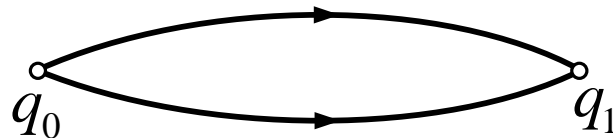


**II)** Interpolation problem.

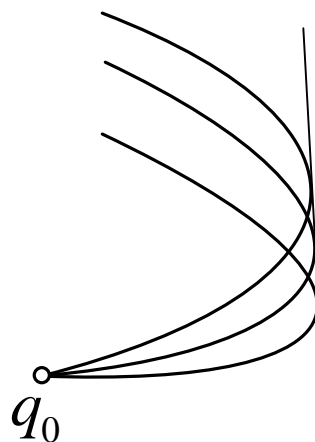
$$M = \mathbb{R}^2 \times S^1, \quad \dot{q} = u_1 f_1 + u_2 f_2.$$

Where to “cut” geodesics?

Maxwell points: at least two geodesics of equal length connect the same points.



Conjugate points: the envelope of the family of geodesics starting from  $q_0$ .

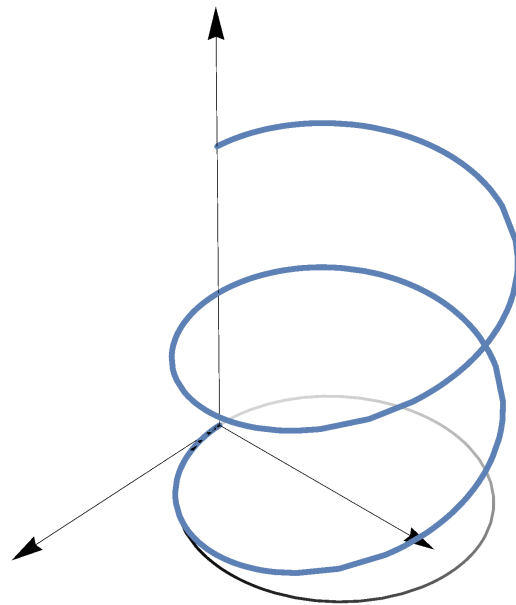


The wave front  $W_{q_0}(r)$  is the set of endpoints of the length  $r$  geodesics starting from  $q_0$ .

The sphere  $S_{q_0}(r)$  is a part of the wave front.

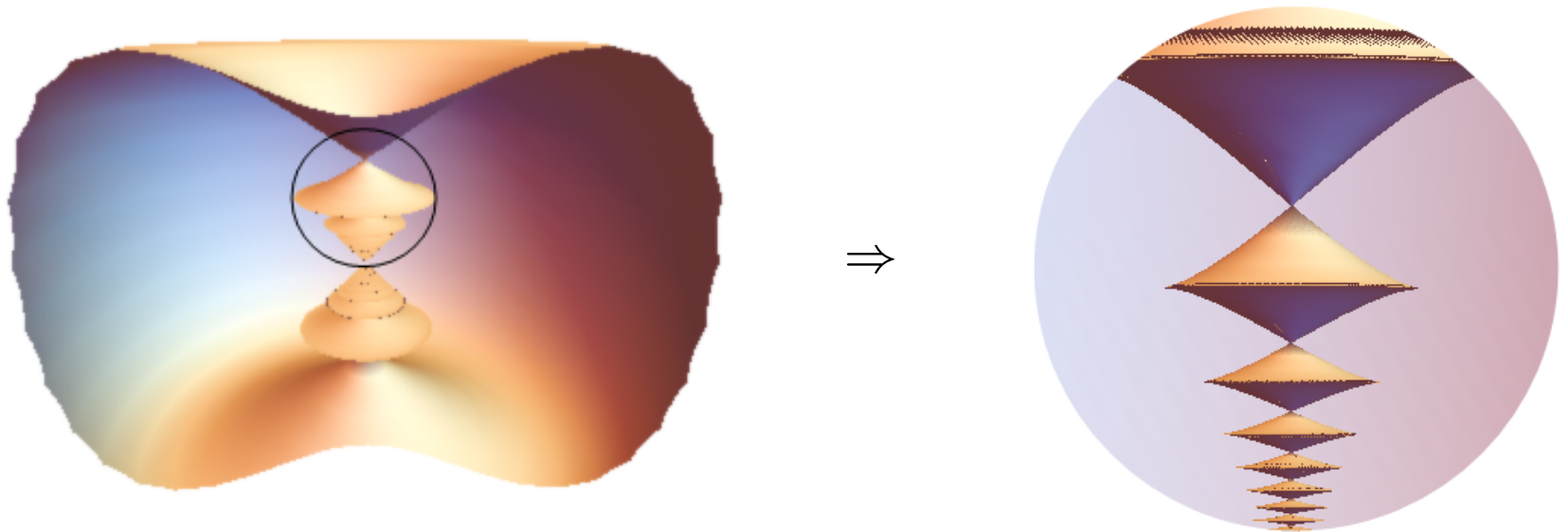
## Dido problem.

Geodesics:

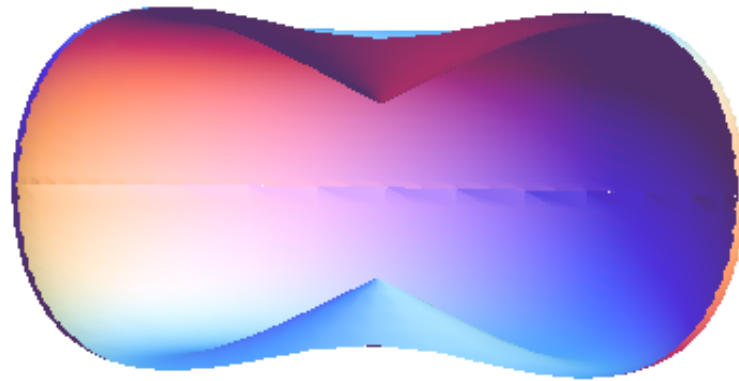


They are characterized by the initial velocity and vertical acceleration; greater acceleration, more tight is the spiral.

The wave front:



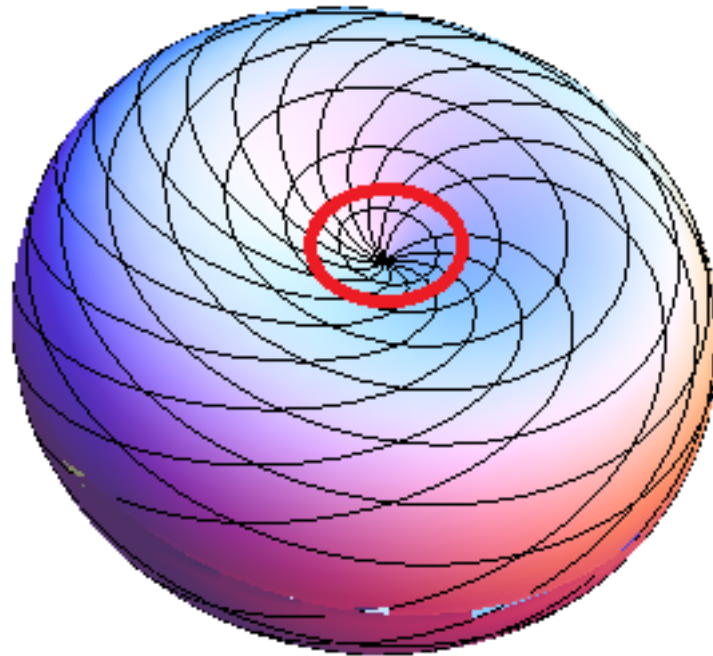
The sphere:



First Maxwell and conjugate points coincide and belong to the vertical line due to the rotational symmetry.

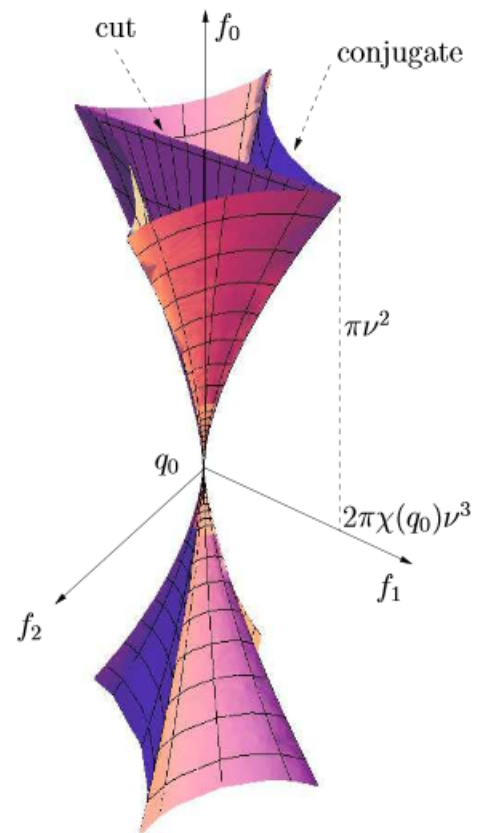
## Breaking the symmetry.

Look under the microscope:

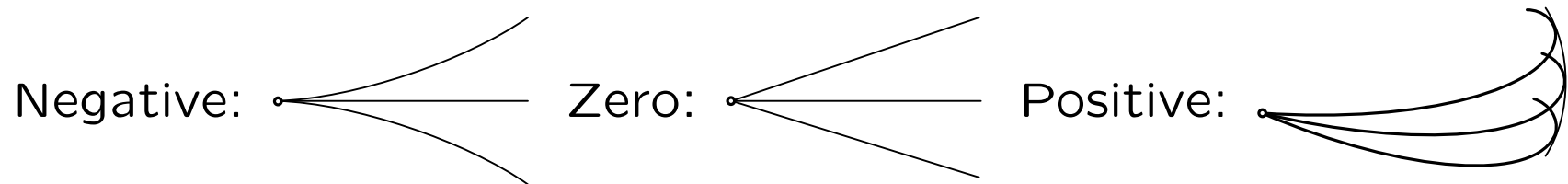


Symmetric  $\Rightarrow$  Generic

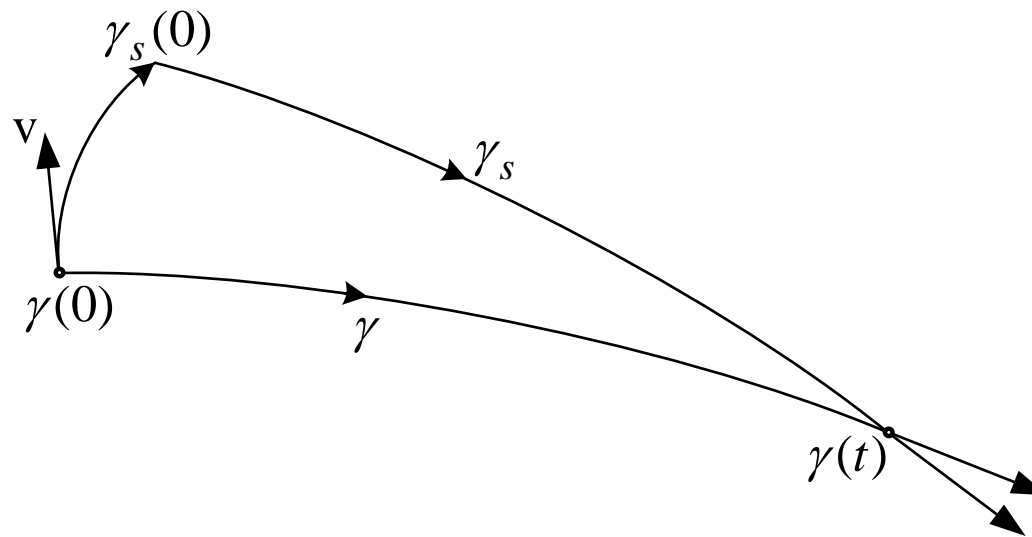
First conjugate and cut loci:



**The curvature** measures deviation from the Euclidean metric along a geodesic.



**Construction.** Given a geodesic  $\gamma$ , horizontal tangent vector  $v \perp \dot{\gamma}(0)$  and  $t > 0$ , consider a family of geodesics:



The curvature:

$$R_\gamma(v) = \frac{\partial^2}{\partial t^2} \frac{\partial^2}{\partial s^2} \left( \frac{t^2}{2} |\dot{\gamma}_s(t) - \dot{\gamma}(t)|^2 \right) \Big|_{t=s=0}.$$

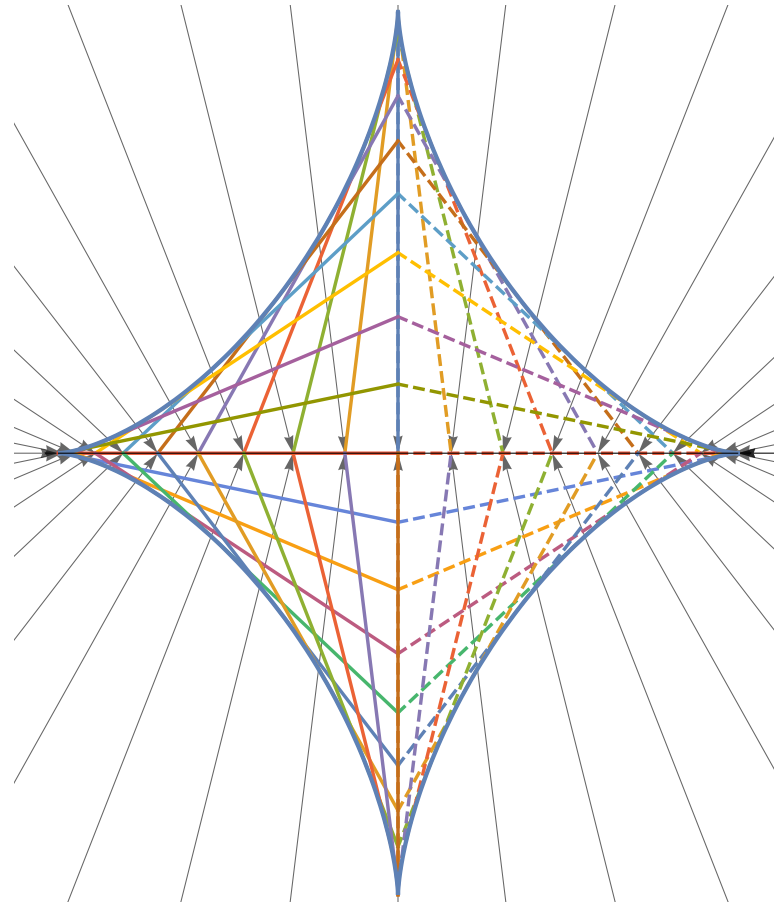
$R_\gamma$  is a quadratic form of  $(k-1)$  variables where  $k$  is the dimension of the horizontal subspace. If  $k$  equals 3, then  $R_\gamma \in \mathbb{R}$ .

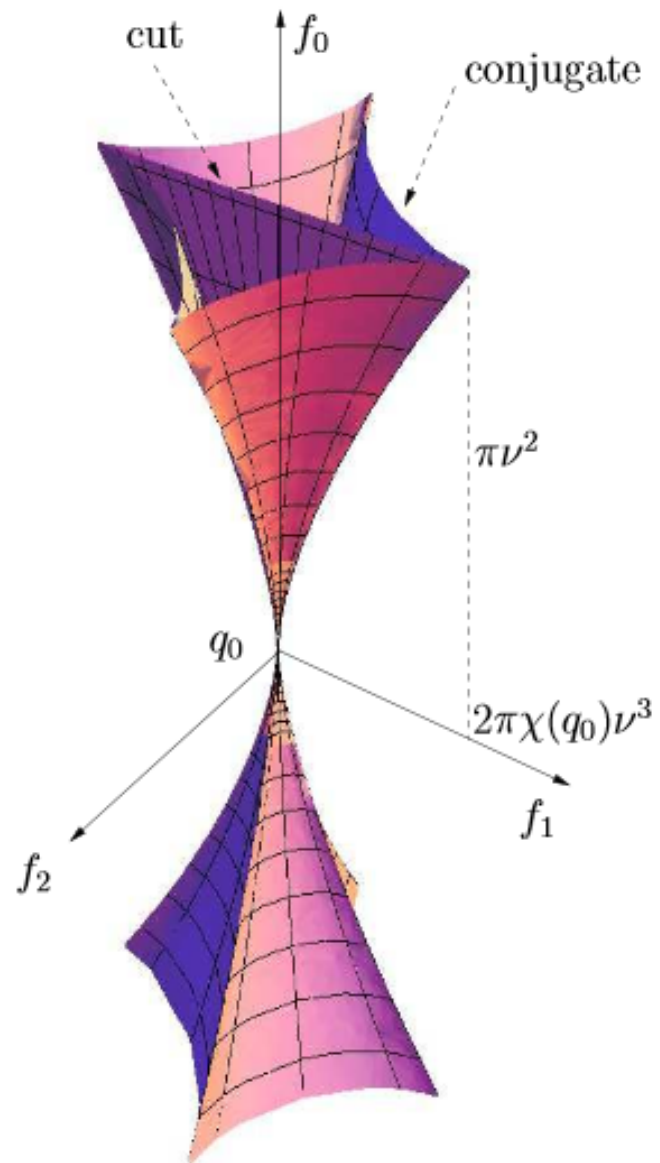
**3-dim problems.** Let  $\nu$  be the initial vertical acceleration of  $\gamma$ ; then

$$R_\gamma = \nu^2 + Q(\dot{\gamma}(0)),$$

where  $Q$  is a quadratic form.  $Q = 0 \Leftrightarrow$  we are locally isometric to the Dido problem.

Maxwell segments are eigenvectors of  $Q$ ; the visible on the sphere first Maxwell segment corresponds to the bigger eigenvalue.





Number  $\chi(q_0)$  is the difference between the eigenvalues of  $Q$ .