Quadratic Cohomology

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Abstract

We study homological invariants of smooth families of real quadratic forms as a step towards a " Lagrange multipliers rule in the large" that intends to describe topology of smooth vector functions in terms of scalar Lagrange functions.

1 Introduction

Morse theory connects homology of Lebesgue sets and level sets of smooth real functions with critical points of the functions. The theory is based on a simple observation that a continuous deformation of the function does not influence the homotopy type of the level and Lebesgue sets for a prescribed value of the function as long as the value is not critical. Moreover, homology of the Lebesgue set is easier to control than one of the level set.

The same observation holds for level sets of smooth vector-functions. A natural generalization of a Lebesgue set is the space of solutions to a system of inequalities. The study of systems of inequalities and equations is partially reduced to the real functions case by the Lagrange multipliers rule. Lagrange function of a vector-function $(\phi^1, \ldots, \phi^k)$ is a linear combination $p_1\phi^1 + \cdots + p_k\phi^k$, $\sum_{i=1}^{k} p_i^2 = 1$, where the coefficients $p_1, \ldots, p_k$ of the linear combination are treated as extra variables, the *Lagrange multipliers*. Vector $0 \in \mathbb{R}^k$ is a critical value of $(\phi^1, \ldots, \phi^k)$ if and only if $0 \in \mathbb{R}$ is a critical value of the Lagrange function.

The title of the famous Marston Morse's book is "The calculus of variations in the large". This paper is a step towards a Lagrange multipliers rule

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in the large. Our first observation, a starting point of the whole story, is that linearity with respect to the Lagrange multipliers is not important. More precisely, if two Lagrange functions

$$f_0(p_1, \ldots, p_k, x) = \sum_{i=1}^{k} p_i \phi_0^i(x), \quad f_1(p_1, \ldots, p_k, x) = \sum_{i=1}^{k} p_i \phi_1^i(x)$$

are connected by a homotopy f_t, $t \in [0,1]$, where f_t are just smooth, not necessary linear with respect to the Lagrange multipliers and 0 is not a critical value of f_t for all $t \in [0,1]$, then zero level sets of the vector functions $(\phi_0^1, \ldots, \phi_0^k)$ and $(\phi_1^1, \ldots, \phi_1^k)$ have equal homologies.

A similar property is valid for systems of inequalities; in this case Lagrange multipliers are taken from the intersection of the sphere with a convex cone. One inequality (like in the Morse theory) corresponds to a point of the sphere. Actually, any point of the sphere of Lagrange multipliers represents a real function. We can think on usual homology of the space of solutions to the inequality as a kind of generalized cohomology of the point (different points may have different generalized cohomologies!). Similarly, generalized cohomology of a convex subset of the sphere is usual homology of the space of solutions to the correspondent system of inequalities. It is easy to extend the construction to more general subsets of the sphere like submanifolds with borders and corners. For the generalized cohomology to have good properties we impose some regularity conditions. In particular, not all convex subsets of the sphere are available but only those corresponding to regular systems of inequalities.

The generalized cohomology satisfies a natural modification of the Eilenberg–Steenrod axioms. The most important "homotopy axiom" is based on the described above property of the homologies of level sets to endure regular homotopies of the Lagrange functions.

Such a cohomology theory is determined by the space of function $span\{\phi^1, \ldots, \phi^k\}$; different spaces of functions give different generalized cohomologies. Moreover, as soon as a space of functions and the axioms are fixed we may try to find other cohomology theory that satisfies the same axioms but may be easier to compute. Such a theory should anyway have an intimate relation to the systems of inequalities and equations. The axioms imply that the cohomology of a point equals usual homology of space of solutions to the correspondent inequality; moreover, the cohomology of a convex set vanishes if the correspondent system of inequalities has no solutions.

This general setting is described in Sections 2–4 of the paper. Main results are presented in Sections $5, 6$, where we build a cohomology theory that satisfies all the axioms in the case when the space of functions is the space of quadratic forms. To compute the cohomology we define a spectral sequence E^r (see Section 5) with clear explicit expressions for all the differentials. The homotopy invariance is proved in Section 6; the proof is based on the results of paper [1].

The page E^2 and the differential d_2 of the spectral sequence E^r are equal to the page F^2 and the differential d_2 of the described in [2] spectral sequence F^r. The sequence F^r converges to the homology of the space of solutions to the system of quadratic inequalities. We do not know higher differentials of the sequence F^r and, for the moment, we do not see a reason for two spectral sequences to be equal. Anyway, this question remains open. The following example shows a flavor of the developed theory. It demonstrates well the geometric meaning of the differential d_3 of the spectral sequence E^r.

Let us consider a 3-dimensional space $isu(2)$ of Hermitian $2 \times 2-$matrices with zero trace. An Hermitian $2 \times 2-$matrix can be treated as a symmetric real $4 \times 4-$matrix commuting with the multiplication of the vectors in $\mathbb{C}^2 = \mathbb{R}^4$ by the imaginary unit i. Thus $isu(2) \subset Sym(\mathbb{R}^4)$, where $Sym(\mathbb{R}^4)$ is a 10-dimensional space of real symmetric $4 \times 4-$matrices. Given a matrix $S \in Sym(\mathbb{R}^4)$, let $\lambda_1(S) \geq \lambda_2(S) \geq \lambda_3(S) \geq \lambda_4(S)$ be its eigenvalues. If $S \in isu(2)$, then $\lambda_1(S) = \lambda_2(S) = -\lambda_3(S) = -\lambda_4(S)$, i.e. the eigenvalues are double (the eigenspaces are complex lines). Recall that, in general, for an eigenvalue to be double is a codimension 2 property in $Sym(\mathbb{R}^4)$

Now take $S_0 \in Sym(\mathbb{R}^4)$ and translate the subspace $isu(2)$ by S_0. We obtain an affine subspace $S_0 + isu(2) \subset Sym(\mathbb{R}^4)$. Matrices from this affine subspace are not obliged to be Hermitian and the eigenvalues are not necessary double. We set:

$$C_j^{S_0} = \{H \in isu(2) : \lambda_j(S_0 + H) = \lambda_{j+1}(S_0 + H)\}, \quad j = 1, 2, 3.$$

For generic S_0, $C_j^{S_0}$ are smooth real algebraic curves in the 3-dimensional space $isu(2)$.

Proposition. $C_j^{S_0}$, $j = 1, 2, 3$, *are not empty. Moreover, for generic S_0, the curve $C_2^{S_0}$ has odd linking numbers with $C_1^{S_0}$ and with $C_3^{S_0}$.*

This proposition is proved in Section 7.

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2 Regular Homotopy

Let M be a smooth compact manifold. Given $\phi^0, \phi^1 \ldots, \phi^k \in C^1(M)$, the system of equations $\phi^0(x) = \cdots = \phi^k(x) = 0$ is *regular* if 0 is not a critical value of the map
$$\varphi = (\phi^0, \ldots, \phi^k)^T : M \to \mathbb{R}^{k+1}.$$

A homotopy $\varphi_t = (\phi_t^0, \ldots, \phi_t^k)^T$ is an *isotopy* of the system of equations $\phi_t^0 = \cdots = \phi_t^k = 0$ if 0 is not a critical value of φ_t, $\forall t \in [0,1]$.

According to the standard Thom lemma, for any isotopy φ_t there exists a family of diffeomorphisms $\Phi_t : M \to M$, $\Phi_0 = id$, such that

$$\varphi_t^{-1}(0) = \Phi_t \left(\varphi_0^{-1}(0) \right), \ \forall t \in [0,1].$$

This is why one uses the term "isotopy". In particular, $\varphi_1^{-1}(0) \cong \varphi_0^{-1}$, $M \setminus \varphi_1^{-1} \cong M \setminus \varphi_0^{-1}$.

Now consider the function $\varphi^* : S^k \times M \to \mathbb{R}$ defined by the formula $\varphi^*(p, x) = \langle p, \varphi(x) \rangle$, where $p \in S^k = \{p \in \mathbb{R}^{k+1} : |p| = 1\}$. It is easy to see that 0 is a critical value of φ if and only if it is a critical value of φ^*.

Nothing prevents us to take any function $f \in C^1(S^k \times M)$. We say that f is regular if 0 is not a critical value of f. A homotopy f_t, $t \in [0,1]$, such that all f_t are regular we call a *regular homotopy*. We have much more regular homotopies than isotopies. Nevertheless regular homotopy preserves an important information on the space of solutions to the system of equations.

Proposition 1. *Assume that f_t is a regular homotopy and $f_0 = \varphi_0^*$, $f_1 = \varphi_1^*$. Then $M \setminus \varphi_0^{-1}(0)$ is homotopy equivalent to $M \setminus \varphi_1^{-1}(0)$.*

Proof. We set

$$B_t = \left\{ (p, x) \in S^k \times M : f_t(p, x) > 0 \right\}.$$

Note that the projections $(p, x) \mapsto x$ restricted to B_0 and B_1 are fiber bundles over $M \setminus \varphi_0^{-1}(0)$ and $M \setminus \varphi_1^{-1}(0)$ whose fibers are hemispheres. In particular, B_0 is homotopy equivalent to $M \setminus \varphi_0^{-1}(0)$ and B_1 is homotopy equivalent to $M \setminus \varphi_1^{-1}(0)$.

Lemma 1. *There exists a smooth family of diffeomorphisms $F_t : S^k \times M \to S^k \times M$ such that $F_0 = id, F_t(B_0) \subset B_t$, $\forall t \in [0,1]$.*

Proof of the lemma. We set $z = (p, x) \in S^k \times M$ and look for a nonautonomous vector field $Z_t(z)$ such that the flow F_t generated by the differential equation $\dot{z} = Z_t(z)$ has the desired property. It is sufficient to find a field Z_t such that the equality $f_t(z) = 0$ implies $\langle d_z f_t, Z_t(z) \rangle > 0$. Moreover, it is sufficient to find such a field locally and then glue local pieces together by a partition of unity. It remains to mention that we can easily do it locally since 0 is not a critical value of f_t. $\square$

Lemma 1 implies that B_0 and B_1 are homotopy equivalent. Indeed, we can make a time substitution $t \mapsto 1 - t$ and find a flow $G_t : S^k \times M \to S^k \times M$ such that $G_t(B_1) \subset B_{1-t}$, $G_0 = id$. The maps $G_1 \circ F_1 : B_0 \to B_0$ and $F_1 \circ G_1 : B_1 \to B_1$ are obviously homotopic to the identity. $\square$

Now I would like to extend the just described construction to systems of inequalities. As we'll see very soon, inequalities are very useful and helpful even if we are mainly interested in the equations. Let $K \subset \mathbb{R}^{k+1}$ be a closed convex cone. A system of inequalities is a relation $\varphi(x) \in K$, $x \in M$, were, as before, $\varphi = (\phi^0, \ldots, \phi^k)^T$. We say that the system of inequalities is *regular* (in the strong sense) if $\operatorname{im} D_x \varphi + K = \mathbb{R}^{k+1}$, $\forall x \in \varphi^{-1}(K)$.

We take the dual cone $K^\circ = \{p \in \mathbb{R}^{k+1} : \langle p, y \rangle \leq 0, \ \forall y \in K\}$ and consider the "manifold with a convex border" $(K^\circ \cap S^k) \times M$. We say that a subset V of a smooth manifold is a *manifold with a convex border* if V is covered by coordinate neigborhoods whose intersections with V are diffeomorphic to closed convex subsets of the Euclidean space. Smooth functions on the manifold with a convex border are restrictions of smooth functions on the ambient manifold. The tangent cone $T_v V$ is the closer of the set of velocities at v of starting from v and contained in V smooth curves.

Let $f : V \to \mathbb{R}$ be a C^1 function. We say that $v \in V$ is a critical point of f if $\langle d_v f, \xi \rangle \leq 0$, $\forall \xi \in T_v V$.

Lemma 2. *If the system of inequalities $\varphi(x) \in K$ is regular (in the strong sense), then 0 is not a critical point of $\varphi^*\big|_{(K^\circ \cap S^k) \times M}$.*

The proof is a straightforward check based on the duality $K^{\circ\circ} = K$; we leave it to the reader. The inverse statement is not true mainly due to the fact that $T_y K$ is, in general, bigger than K.

Definitions of regular functions on the manifold with a convex border and of regular homotopy for such functions are verbal repetitions of the definitions for the manifold without border. An obvious modification of the proof of Proposition 1 gives:

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Proposition 2. *Assume that* $f_t : (K^\circ \cap S^k) \times M \to \mathbb{R}$, $t \in [0,1]$, *is a regular homotopy and* $f_0 = \varphi_0^*|_{(K^\circ \cap S^k) \times M}$, $f_1 = \varphi_1^*|_{(K^\circ \cap S^k) \times M}$. *Then* $M \setminus \varphi_0^{-1}(K)$ *is homotopy equivalent to* $M \setminus \varphi_1^{-1}(K)$.

Remark. Actually, the (obviously modified) proof of Proposition 1 gives more; namely, under conditions of Proposition 2 the inclusion

$$(t, B_t) \hookrightarrow \bigcup_{\tau \in [0,1]} (\tau, B_\tau)$$

of the subspaces of $[0,1] \times V \times M$ is a homotopy equivalence, $\forall\, t \in [0,1]$.

So the homotopy type of the complement to the space of solutions to the system of inequalities is preserved by regular homotopies. It happens that homology of the space of solutions is preserved as well.

Proposition 3. *Assume that* $f_t : (K^\circ \cap S^k) \times M \to \mathbb{R}$, $t \in [0,1]$, *is a regular homotopy and* $f_0 = \varphi_0^*|_{(K^\circ \cap S^k) \times M}$, $f_1 = \varphi_1^*|_{(K^\circ \cap S^k) \times M}$. *Then the homology groups of* $\varphi_0^{-1}(K)$ *and* $\varphi_1^{-1}(K)$ *with coefficients in a field are isomorphic.*

Proof. We start from the case $K \neq -K$, i.e. K is not a subspace and the system of inequalities is not just a system of equations. In this case, $K^\circ \cap S^k$ is contractible and we have the following series of homotopy equivalences of the pairs:

$$\left(M, M \setminus \varphi_0^{-1}(K) \right) \sim \left((K^\circ \cap S^k) \times M, B_0 \right) \sim$$
$$\left((K^\circ \cap S^k) \times M, B_1 \right) \sim \left(M, M \setminus \varphi_1^{-1}(K) \right),$$

where $B_t = \left\{ (p, x) \in (K^\circ \cap S^k) \times M : f_t(p, x) > 0 \right\}$ (see the proof of Proposition 1). Hence $H^*(M, M \setminus \varphi_0^{-1}(K)) \cong H^*(M, M \setminus \varphi_1^{-1}(K))$. The Alexander–Pontryagin duality completes the proof.

The case of a system of equations is easily reduced to the just studied case if we add the tautological inequality $1 \geq 0$ to the system. Let us explain this in more detail. If K is a subspace, then we may assume without lack of generality that $K = 0$. Now extend the function f_t to $\mathbb{R}^{k+1} \times M$ as a degree one homogeneous function with respect to the variable p (keeping the symbol f_t for the extension) and consider the functions

$$\bar{f}_t : (p, \nu, x) \mapsto f_t(p, x) + \nu, \quad |p|^2 + \nu^2 = 1, \ \nu \leq 0.$$

It is easy to see that $\bar{f}_t$ are regular. To be absolutely rigorous, we have to smooth out f_t at the points $(0, x)$ but, in fact, nothing depends on the way we do it because $\bar{f}_t$ is far from 0 at these points. $\qquad \square$

3 Localization

Let V be a manifold with a convex border and $f : V \times M \to \mathbb{R}$ a C^1-function. In this section, we assume that M is a real-analytic manifold and $f(v, \cdot)$ is a subanalytic function, $\forall\, v \in V$. It is convenient to think about f as a family of subanalytic functions $f(v, \cdot)$ on M which depends on the parameter $v \in V$, and we introduce the notation $f_v \doteq f(v, \cdot)$. "Localization" in this section is the localization with respect to the parameter v; the variable $x \in M$ remains global.

We say that the family f_v is regular at $v_0 \in V$ if the set $\{v_0\} \times f_{v_0}^{-1}(0)$ does not contain critical points of f.

Proposition 4. *Assume that the family f_v, $v \in V$, is regular at $v_0 \in V$. Then v_0 has a compact neighborhood O_{v_0} and centered at v_0 local coordinates Φ such that $U_0 \doteq \Phi(O_{v_0})$ is convex and the function $(f \circ \Phi + t)\big|_{\varepsilon U_0 \times M}$ is regular for any sufficiently small nonnegative constants t, ε one of which is strictly positive.*

Proof. We may assume that $v_0 = 0$ is the origin of a Euclidean space and $\Phi = id$. Given $a \in C^1(M)$, $y \in \mathbb{R}$, we set $C_a(y) = \{x \in a^{-1}(y) : d_x a = 0\}$ If $0 \in \mathbb{R}$ is not a critical value of f_0, i.e. $C_{f_0} = \emptyset$, then the statement is obvious; otherwise, for any $x \in C_{f_0}$ there exists $\nu_x \in U_0$ such that $\left\langle \frac{\partial f}{\partial v}(0, x), \nu_x \right\rangle \geq \alpha > 0$, where α is a positive constant. Then, by the continuity, there exists $\delta > 0$ such that for any $\tau \in [-\delta, \delta]$, $v \in \delta U_0$, $x \in C_{f_v}(\tau)$, there exists $\hat{x} \in C_{f_0}(0)$ such that

$$\left\langle \frac{\partial f}{\partial v}(v, x), \nu_{\hat{x}} \right\rangle \geq \delta > 0 \tag{1}$$

Now let $v \in \varepsilon U_0$, $t \in [0, \delta]$, and $x \in C_{f_v}(-t)$; then $d_x f_v = 0$ and $|d_x f_0| \leq c\varepsilon$ for some constant c. We have:

$$f(0, x) = f(v, x) - \left\langle \frac{\partial f}{\partial v}(v, x), v \right\rangle + o(\varepsilon),$$

where $\frac{o(\varepsilon)}{\varepsilon} \to 0$ as $\varepsilon \to 0$ uniformly for all $v \in \varepsilon U_0$, $x \in C_{f_0}(-t)$, $t \in [0, \delta]$. Then

$$\left\langle \frac{\partial f}{\partial v}(v, x), -v \right\rangle = t + f_0(x) - o(\varepsilon)$$

and, according to (1),

$$\left\langle \frac{\partial f}{\partial v}(v, x), \varepsilon \nu_{\hat{x}} - v \right\rangle \geq t + f_0(x) + \varepsilon \delta - o(\varepsilon).$$

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The Lojasevic inequality [3] gives:

$$|f_0(x)| \leq c'|d_x f_0|^{1+\rho} \leq c' c^{1+\rho} \varepsilon^{1+\rho},$$

where c', ρ are positive constants. Hence $\left\langle \frac{\partial f}{\partial v}(v, x), \varepsilon \nu_{\hat{x}} - v \right\rangle > 0$ if ε is sufficiently small. $\qquad \square$

Corollary 1. *Let V be a compact convex set, $0 \in V$. Assume that the family f_v, $v \in V$, is regular at 0. Then for any sufficiently small $\varepsilon > 0$ the homotopy*

$$(t, v, x) \mapsto f(tv, x) + (1-t)\varepsilon, \quad t \in [0, 1], \ v \in \varepsilon V, \ x \in M.$$

between $f\big|_{\varepsilon V \times M}$ and the constant family $(v, x) \mapsto f(0, x) + \varepsilon$ is regular.

4 A Cohomology Theory

Let M be a real-analytic manifold and $\mathcal{A} \subset C^1(M)$ a set of subanalytic functions. Let $W \subset V$ be a pair of manifolds with convex borders and $f : V \times M \to \mathbb{R}$ a regular function such that $f_v \in \mathcal{A}$, $\forall v \in V$, and $f\big|_{W \times M}$ is also regular.

We set $B_f = \{(v, x) : v \in V, \ f(v, x) > 0\}$ and define

$$H_{\mathcal{A}}^{\cdot}(f_V, f_W) \doteq H^{\cdot}(V \times M, (W \times M) \cup B_f), \quad H_{\mathcal{A}}^{\cdot}(f) \doteq H_{\mathcal{A}}^{\cdot}(f_V . f_\emptyset).$$

The pairs of regular functions (f_V, f_W) form a category $\mathfrak{F}_{\mathcal{A}}$ with morphisms $\varphi^* : (f_{V_0}^0, f_{W_0}^0) \mapsto (f_{V_1}^1, f_{W_1}^1)$, where $\varphi : V_1 \to V_0$ is a C^1-map such that $\varphi(W_1) \subset W_0$ and $f_v^1 = f_{\varphi(v)}^0$, $\forall v \in V_1$. Then $H_{\mathcal{A}}^{\cdot}$ is a functor from this category to the category of commutative groups.

This is a kind of cohomology functor which satisfies natural modifications of the Steenrod–Eilenberg axioms except of the dimension axiom. The exactness and excision are obvious and we do not repeat them. Homotopy axiom deals with $f : [0, 1] \times V \times M \to \mathbb{R}$ such that $f_{\{t\} \times V} \in \mathfrak{F}_{\mathcal{A}}, \forall t \in [0, 1]$, and claims that the inclusions $\{t\} \times V \hookrightarrow [0, 1] \times V, \ t \in [0, 1]$, induce the isomorphisms of cohomology groups:

$$H_{\mathcal{A}}^{\cdot}\left(f_{[0,1] \times V}, f_{[0,1] \times W}\right) \cong H_{\mathcal{A}}^{\cdot}\left(f_{\{t\} \times V}, f_{\{t\} \times W}\right).$$

This simple but not totally obvious fact was explained in Section 2.

The dimension axiom is substituted by the following one: if $V = \{v\}$ is a point then
$$H_{\mathcal{A}}^{\cdot}\left(f_{\{v\}}\right) = H^{\cdot}\left(M, \{x \in M : f_v(x) > 0\}\right).$$
The "points" for us are regular elements of $\mathcal{A}$ and different points may have different cohomology.

Standard singular cohomology is a special case. Indeed, let the set $\mathcal{A}$ consist of one point, $\mathcal{A} = \{a\}$, and $a(x) < 0$, $\forall\, x \in M$. We have:
$$H_{\{a\}}^{\cdot}(V, W) = H^{\cdot}(V, W) \times H^{\cdot}(M).$$

Now assume that $\mathcal{A} + t \subset \mathcal{A}$ for any nonnegative constant t. Given a map $v \mapsto f_v$ from V to $\mathcal{A}$ we denote by $(f + t)_{[0,c] \times V}$ the map $(t, v) \mapsto f_v + t$, $t \in [0, c]$, $v \in V$. It was proved in Section 3 that for any $v \in V$ there exists a neigborhood $U_v \subset V$ and $\varepsilon > 0$ such that the inclusions
$$U_v \times \{0\} \hookrightarrow U_v \times [0, \varepsilon], \quad \{v\} \times \{\varepsilon\} \hookrightarrow U_v \times [0, \varepsilon]$$
induce the isomorphisms of the cohomology groups
$$H_{\mathcal{A}}^{\cdot}\left(f_{U_v}\right) \cong H_{\mathcal{A}}^{\cdot}\left((f + t)_{[0,\varepsilon] \times U_v}\right) \cong H_{\mathcal{A}}^{\cdot}\left(f_{\{v\}} + \varepsilon\right).$$

In other words, cohomology of a "small neighborhood" is equal to the cohomology of a "point".

Now assume that the cohomology are taken with coefficients in a field and that $\dim M = n$. Then the cohomology of a "point"
$$H_{\mathcal{A}}^i\left(f_{\{v\}}\right) = H^i\left(M, \{x \in M : f_v(x) > 0\}\right) = H_{n-i}\left(\{x \in M : f_v(x) \le 0\}\right),$$
$0 \le i \le n$, is simply usual homology of the space of solutions to the inequality $f_v(x) \le 0$.

The localization at a point plus the based on the axioms algebraic homology machinary gives a good chance to recover usual homology of the space of solutions of a system of inequalities from ones of the individual inequalities of the form $a(x) \le 0$, where $a \in \mathcal{A}$. The success is somewhow guaranteed if $H_{\mathcal{A}}^{\cdot}$ is a unique cohomology theory for $\mathcal{A}$ that satisfies the described axioms. On the other hand, any other cohomology theory that satisfies the same axioms gives additional important invariants of systems of inequalities or equations for functions from $\mathcal{A}$.

Let me explain it better for regular systems of equations

$$\phi^0(x) = \cdots = \phi^k(x) = 0, \quad \phi^i \in \mathcal{A}, \ i = 0, 1, \ldots, k.$$

An isotopy $\varphi_t = (\phi_t^0, \ldots, \phi_t^k)^T$, $t \in [0, 1]$, of such systems is a $\mathcal{A}$-*rigid isotopy* if $\phi_t^i \in \mathcal{A}$ for all $t \in [0, 1]$. In this case, $\varphi_t^* \in \mathfrak{F}_\mathcal{A}$, where, recall,

$$\varphi_t^* : S^k \times M \to \mathbb{R}, \quad \varphi_t^*(p, x) = \langle p, \varphi_t(x) \rangle.$$

Let $\hat{H}_\mathcal{A}$ be a cohomology functor that satisfies our axioms; then, according to the homotopy axiom, $\hat{H}_\mathcal{A}(\varphi_0^*) = \hat{H}_\mathcal{A}(\varphi_1^*)$. In other words, $\hat{H}_\mathcal{A}$ is an invariant of the $\mathcal{A}$-rigid isotopy. Moreover, it is an invariant of regular homotopies in $\mathfrak{F}_\mathcal{A}$ that are much more general than $\mathcal{A}$-rigid isotopies.

Let $\varphi = (\phi^0, \ldots, \phi^k)^T$, $(\nu, p) \in \mathbb{R} \times \mathbb{R}^{k+1}$, $x \in M$; we set $\bar{\varphi}^*(\nu, p, x) = \nu + \langle p, \varphi(x) \rangle$ and denote by S_-^{k+1} the low semi-sphere in $\mathbb{R} \times \mathbb{R}^{k+1}$, $S_-^{k+1} = \{(\nu, p) : \nu \leq 0, \ \nu^2 + |p|^2 = 1\}$.

Proposition 5. *If* $\varphi^{-1}(0) = \emptyset$, *then* $\hat{H}_\mathcal{A}\left(\bar{\varphi}_{S_-^{k+1}}^*\right) = 0$.

Proof. Let $c \in \mathbb{R}$, $B_c^{k+1} = \{(c, p) : p \in \mathbb{R}^{k+1}, \ |p| \leq 1\}$. Note that $\bar{\varphi}^*\big|_{B_c^{k+1} \times M}$ is a regular function for any $c > 0$ (this is true for any smooth map $\varphi : M \to \mathbb{R}^{k+1}$). Moreover, $\bar{\varphi}_{B_c^{k+1}}^*$ is regularly homotopic in $\mathfrak{F}_\mathcal{A}$ to the constant function c; indeed, the homothety of the ball B_c^{k+1} to its center along the radii provides us with the desired regular homotopy. Hence $\hat{H}_\mathcal{A}\left(\bar{\varphi}_{B_c^{k+1}}^*\right) = 0$.

The function $\bar{\varphi}^*\big|_{B_0^{k+1} \times M}$ is regular if and only if $\varphi^{-1}(0) = \emptyset$. If it is regular, then it is regularly homotopic to $\bar{\varphi}^*\big|_{B_c^{k+1} \times M}$, where $c > 0$, and $H_\mathcal{A}\left(\bar{\varphi}_{B_0^{k+1}}^*\right) = 0$. It remains to note that the homotopy between $\bar{\varphi}_{B_0^{k+1}}^*$ and $\bar{\varphi}_{S_-^{k+1}}^*$ induced by the homotopy $(t; \nu, p) \mapsto ((1 - t)\nu, p)$, $t \in [0, 1]$, $(\nu, p) \in S_-^{k+1}$, is also regular. $\square$

Let $M = \mathbb{R}P^N = \{(x, -x) : x \in S^k\}$ and $\mathcal{Q}(N)$ the space of real quadratic forms on $\mathbb{R}^{N+1}$ treated as functions on $\mathbb{R}P^N$. The main goal of this paper is to construct a cohomology theory $\hat{H}_{\mathcal{Q}(N)}$. This is not just an abstract construction: we give an effective way to compute the cohomology.

In what follows all cochains and cohomologies are with coefficients in $\mathbb{Z}_2$. We omit symbol $\mathbb{Z}_2$ to simplify notations.

5 Spectral Sequence

Now we focus on the space $\mathcal{Q}(N)$ with fixed N and omit the argument N in order to simplify notations. We denote by the same symbol a quadratic form on $\mathbb{R}^{N+1}$ and the induced by this form function on $\mathbb{R}P^N$. A quadratic form q induces a regular function on $\mathbb{R}P^N$ if and only if $\ker q = 0$. More precisely, critical points of $q : \mathbb{R}P^N \to \mathbb{R}$ at $q^{-1}(0)$ are exactly $\bar{x} = (x, -x) \in \mathbb{R}P^N$ such that $x \in \ker q \cap S^N$.

Some notations. Let $\lambda_1(q) \geq \cdots \geq \lambda_{N+1}(q)$ be the eigenvalues of the symmetric operator associated to the quadratic form $q \in \mathcal{Q}$. We set

$$\Lambda_{j,m} = \{q \in \mathcal{Q} : \lambda_{j-1}(q) \neq \lambda_j(q) = \lambda_{j+m-1}(q) \neq \lambda_{j+m}(q)\},$$

$$\Lambda_{j,m}^0 = \{q \in \Lambda_{j,m} : \lambda_j(q) = 0\},$$

$j = 1, \ldots, N, \ m = 2, \ldots, N - j + 2$. It is well-known that $\Lambda_{j,m}$ is a smooth submanifold of codimension $\frac{m(m+1)}{2} - 1$ in $\mathcal{Q}$ while $\Lambda_{j,m}^0$ is a codimension 1 submanifold of $\Lambda_{j,m}$ (see [1, Prop. 1] for a short proof).

We say that the pair $(f_V, f_W) \in \mathfrak{F}_\mathcal{Q}$ is in the general position if the borders ∂V, ∂W are smooth and the map $v \mapsto f_v$, $v \in V$, as well as the restrictions of this map to W, ∂V, ∂W are transversal to $\Lambda_{j,m}$ and $\Lambda_{j,m}^0$, for $j = 1, \ldots, N, \ m = 2, \ldots, N - j + 2$.

It is sufficient to construct $\hat{H}(f_V, f_W)$ and check the axioms for the pairs in general position. Indeed, if the the borders ∂V, ∂W are smooth, then standard transversality arguments allow to approximate any pair by a pair in the general position. Moreover, any two sufficiently close approximations are regularly homotopic and have equal cohomology $\hat{H}$ according to the homotopy axiom. The cohomology of the given pair is equal, by the definition, to the cohomology of a sufficiently close approximation in the general position.

Similar arguments work in the case of of nonsmooth borders. Given a manifold V with a convex border we can always find a transversal to the border ∂V smooth vector field. Passing through ∂V trajectories of this field provide us with a tubular neighborhood of the border. Smooth sections of the tubular neighborhood give us smooth approximations of ∂V inside V and we obtain $\tilde{V} \subset V$, where $\partial \tilde{V}$ is a smooth approximation of ∂V. The approximation is good if time to move from $\partial \tilde{V}$ to ∂V along trajectories of our transversal vector field is a C^0-small semi-concave function with a uniformly bounded differential (recall that the differential is defined almost everywhere).

It is easy to see that $(f_{\tilde{V}}, f_{\tilde{W}}) \in \mathfrak{F}_{\mathcal{Q}}$ for any sufficiently good approximation $\tilde{W} \subset W$, $\tilde{W} \subset \tilde{V} \subset V$. Moreover, natural diffeomorphisms of different tubular neighborhoods induce homotopic to identity diffeomorphisms of good approximations $(\tilde{V}, \tilde{W})$ and natural isomorphisms of cohomologies $\hat{H}(f_{\tilde{V}}, f_{\tilde{W}})$. The cohomology $\hat{H}(f_V, f_W)$ is equal, by definition, to $\hat{H}(f_{\tilde{V}}, f_{\tilde{W}})$, where $(\tilde{V}, \tilde{W})$ is a sufficiently good approximation of (V, W) by the pair of manifolds with smooth borders.

Let $f : V \to \mathcal{Q}$, $f \in \mathfrak{F}_{\mathcal{Q}}$, be in the general position[1] and

$$V_f^j = \{v \in V : \lambda_j(f(v)) > 0\}, \quad j = 1, \dots, N+1,$$

a decreasing filtration of V by open subsets. We equip V with a Riemannian metric and take $\varepsilon > 0$ so small that V_f^j and $f^{-1}(\Lambda_{j,m})$ are homotopy retracts of their radius $(\dim V)\varepsilon$ neighborhood, $j = 1, \dots, N+1$, $m = 2, \dots, N-j+2$.

Now consider a smooth singular simplex $\varsigma : \Delta^i \to V$, where Δ^i is the standard i-dimensional simplex. We say that ς is adapted to f if the diameter of $\varsigma(\Delta^i)$ is smaller than ε and the restriction of $f \circ \varsigma$ to any face D of Δ^i satisfies the following properties:

(i) $f \circ \varsigma|_D \pitchfork \Lambda_{j,m}$;

(ii) if $\dim D = 4$ and $f \circ \varsigma(D) \cap \Lambda_{j,2} \neq \emptyset$ then $f \circ \varsigma(D) \cap \Lambda_{j+1,2} = f \circ \varsigma(D) \cap \Lambda_{j-1,2} = \emptyset$, $j = 1, \dots, N$.

Manifold V admits a triangulation by adapted simpleces. A more delicate property (ii) can be achieved because $\bar{\Lambda}_{j,2} \cap \bar{\Lambda}_{j+1,2} = \bar{\Lambda}_{j,3}$ has codimension 5 in $\mathcal{Q}$.

We denote by $C_{f,i}(V)$ the space of i-dimensional singular chains in V generated by the adapted singular simpleces with coefficients in $\mathbb{Z}_2$. Let U be an open subset of V; then $C_{f,i}(U)$ is a subspace of $C_{f,i}(V)$ generated by singular simpleces with values in U and $C_f^i(V, U)$ is the space of linear forms on $C_{f,i}(V)$ that vanish on $C_{f,i}(U)$. We obtain a cochain complex

$$\dots \to C_f^{i-1}(V, U) \xrightarrow{\delta} C_f^i(V, U) \xrightarrow{\delta} C_f^{i+1}(V, U) \to \dots, \tag{2}$$

where δ is usual coboundary of singular cochains. The cohomology of the complex (2) coincides with standard cohomology of the pair (V, U) with coefficients in $\mathbb{Z}_2$: $\ker \delta / \operatorname{im} \delta = H^\cdot(V, U)$.

We define cocycles $l_f^j \in C_f^2(V)$, $j = 1, \dots, N$ as follows: given a singular simplex $\varsigma \in C_{f,2}(V)$, $l_f^j(\varsigma)$ is the intersection number modulo 2 of $f \circ \varsigma$ and

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[1] for simplicity, we keep symbol f for the map $v \mapsto f_v$

$\bar{\Lambda}_{j,2}$. We have: $l_f^j \smile l_f^{j+1} = 0$, $j = 1, \ldots, N-1$. Here $\smile$ is the cup product of singular cochains. The maps $\ell_j : \varsigma \to \varsigma \smile l_f^j$ define homomorphisms $\ell_j : C_f^i(V,U) \to C_f^{i+2}(V,U)$. We have $\delta \circ \ell_j = \ell_j \circ \delta$, $\ell_j \circ \ell_{j+1} = 0$.

Given $\tau > 0$ let $V_f^j(\tau)$ be the radius τ neighborhood of V_f^j. We set:

$$C_j^i(f) = C_f^i\left(V, V_f^j(i\varepsilon)\right), \quad C^n(f) = \bigoplus_{i+j=n} C_{j+1}^i(f);$$

then $\ell_j\left(C_{j+1}^i(f)\right) \subset C_j^{i+2}(f)$. Finally, we define the differential $d : C^n(f) \to C^{n+1}(f)$ by the formula $d\big|_{C_{j-1}^i}(f) = \delta + \ell_j$.

The cohomology $\hat{H}_{\mathcal{Q}}(f)$ is, by definition, the cohomology of the complex

$$\ldots \to C^{n-1}(f) \xrightarrow{d} C^n(f) \xrightarrow{d} C^{n+1}(f) \to \ldots \tag{3}$$

Remark. A pedantic reader would say that cochain groups $C^n(f)$ depend on the small parameter ε. It is not hard to see that the cohomologies of complex (3) for different ε are naturally isomorphic.

Consider a filtration of the complex $\bigoplus_{n\geq 0} C^n(f) = \bigoplus_{n\geq 0}\bigoplus_{i\geq 0} C_{n-i+1}^i(f)$ by sub-complexes $\bigoplus_{n\geq 0}\bigoplus_{i\geq \alpha} C_{n-i+1}^i(f)$, $\alpha = 0, 1, \ldots, \dim V$ and the spectral sequence $E_{i,j}^r$ of this filtration converging to $\hat{H}_{\mathcal{Q}}(f)$. We have:

$$E_{i,j}^1 = C_{j+1}^i(f), \quad d_1 : C_{j+1}^i(f) \to C_{j+1}^{i+1}(f), \ d_1 = \delta.$$

Hence

$$E_{ij}^2 = H^i(V, V_f^{j+1}), \quad d_2 : H^i(V, V_f^{j+1}) \to H^{i+2}(V, V_f^j). \tag{4}$$

Moreover, the differential (4) is induced by ℓ_j and has a very simple explicit expression. Namely, let $\bar{l}_f^j \in H^2\left(V, V \setminus f^{-1}(\bar{\Lambda}_{j,2})\right)$ be the cohomology class of the cocycle l_f^j. Then d_2 is the composition of the map

$$\bar{\ell}_j : H^i(V, V_f^{j+1}) \to H^{i+2}\left(V, V_f^{j+1} \cup (V \setminus f^{-1}(\bar{\Lambda}_{j,2}))\right)$$

defined by the formula $\bar{\ell}_j(x) = x \smile \bar{l}_f^j$, $x \in H^i(V, V_f^{j+1})$, and the homomorphism $H^{i+2}\left(V, V_f^{j+1} \cup (V \setminus f^{-1}(\bar{\Lambda}_{j,2}))\right) \to H^{i+2}(V, V_f^j)$ induced by the inclusion $V_f^j \subset V_f^{j+1} \cup \left(V \setminus f^{-1}(\bar{\Lambda}_{j,2})\right)$.

We see that $E_{i,j}^2$ and d_2 coincide with the second page $F_{i,j}^2$ and the differential $d_2 : F_{i,j}^2 \to F_{i+2,j-1}^2$ of the converging to $H_{\mathcal{Q}}(f)$ spectral sequence

studied in [2] (see Theorems 25 and 28 of the cited paper). Hence $E_{i,j}^3 = F_{i,j}^3$. Now we are going to give simple explicit expressions for all differentials $d_r : E_{i,j}^r \to E_{i+r,j-r+1}^r$, $r \geq 3$.

Let $\xi \in C_{j+1}^i(f) = E_{i,j}^1$ be a δ-cocycle such that its cohomology class $\bar{\xi} \in H^i(V, V_f^{j+1}) = E_{i,j}^2$ is a d_2-cocycle. Then $\xi \smile l_f^j = \delta\eta$, where $\eta \in C_j^{i+1}(f)$. Moreover, $d_3(\bar{\xi})$ is the cohomology class of $\eta \smile l_f^{j-1}$ in $H^{i+3}(V, V_f^{j-1})$ modulo d_2-coboundaries while $l_f^j \smile l_f^{j-1} = 0$. Hence $d_3(\bar{\xi})$ is the Massey product $\langle \bar{\xi}, \bar{l}_f^j, \bar{l}_f^{j-1} \rangle$ combined with an appropriate inclusion homomorphism (see [4, Ch. 8] for the definition and basic properties of Massey products).

Now assume that ξ survives in $E_{i,j}^r$, i.e. classes of ξ are cocycles for $d_3, \ldots, d_{r-1}$. The induction procedure implies that $d_r(\xi)$ is the r-fold Massey product $\langle \bar{\xi}, \bar{l}_f^j, \ldots, \bar{l}_f^{j-r+2} \rangle$ combined with appropriate inclusion homomorphisms.

Indeed, since the class of ξ is d_{r-1}-cocycle then, according to the induction assumption, $\langle \bar{\xi}, \bar{l}_f^j, \ldots, \bar{l}_f^{j-r+3} \rangle \ni \delta\zeta$, where $\zeta \in C_{j-r+3}^{i+r-2}(f)$, and $d_r(\bar{\xi})$ is the class of $\zeta \smile l_f^{j-r+2}$.

If $\dim V \leq k$, then $E_{i,j}^2 = 0$ for $i > k$. In particular, if $\dim V = 3$ then the last possibly nontrivial differential is d_3. This differential has a clear geometric meaning that we are going to describe. Assume that $H_1(V; \mathbb{Z}_2) = 0$ and ∂V is connected or empty (the three-dimensional sphere and ball are available). Then $H_2(V; \mathbb{Z}_2) = 0$ and the linking number $\bmod\, 2$ of a 1-dimensional cycle in V with a 1-dimensional cycle in $(V, \partial V)$ are well-defined. We have:

$$d_3 : H^0(V, V_f^{j+1}) \longrightarrow H^3(V, V_f^{j-1}). \qquad (*)$$

Moreover, ranks of $H^0(V, V_f^{j+1})$ and $H^3(V, V_f^{j-1})$ are either one or zero.

If both ranks are equal to one, then d_3 sends the generator of $H^0(V, V_f^{j+1})$ to the generator of $H^3(V, V_f^{j-1})$ multiplied by the linking number of 1-dimensional cycles $f^{-1}(\Lambda_{j,2})$ and $f^{-1}(\Lambda_{j-1,2})$, according to the direct implementation of the above construction.

Let $W \subset V$ be such that the pair $(f_V, f_W) \in \mathfrak{F}_{\mathcal{Q}}$ is in the general position and $\tilde{W} \supset W$ be an appropriate tubular neighborhood of W such that the pairs (W, W_f^j) are homotopy retracts of $(\tilde{W}, \tilde{W}_f^j)$ and $\hat{H}^\cdot(f_{\tilde{W}})$ is naturally isomorphic to $\hat{H}^\cdot(f_{\tilde{W}})$. We define:

$$C_j^i(f_V, f_W) \doteq C_j^i(f_V) \cap C_f^i(V, \tilde{W}), \quad C^n(f_V, f_W) = \bigoplus_{i+j=n} C_{j+1}^i(f_V, f_W).$$

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The cohomology $\hat{H}_{\mathcal{Q}}(f_V, f_W)$ is, by definition, the cohomology of the complex

$$\ldots \to C^{n-1}(f_V, f_W) \xrightarrow{d} C^n(f_V, f_W) \xrightarrow{d} C^{n+1}(f_V, f_W) \to \ldots.$$

The excision axiom holds automatically while the obvious exact sequence

$$0 \to C^n(f_V, f_W) \to C^n(f_V) \to C^n(f_{\tilde{W}}) \to 0$$

implies the long exact sequence

$$\cdots \to \hat{H}_{\mathcal{Q}}^n(f_V) \to \hat{H}_{\mathcal{Q}}^n(f_W) \to \hat{H}_{\mathcal{Q}}^{n+1}(f_V, f_W) \to \hat{H}_{\mathcal{Q}}^{n+1}(f_V) \to \cdots .$$

If $V = \{v\}$ is a point, then $\hat{H}_{\mathcal{Q}}(f_{\{v\}}) = H_{\mathcal{Q}}(f_{\{v\}})$ since the spectral sequence $E_{i,j}^r$ degenerates in the page $E_{i,j}^2$ in this case.

The homotopy property is automatic for homotopies in the class of functions in the general position. This property is not at all trivial for homotopies that include functions not in the general position. Moreover, this property is, actually, central point of the whole story; we prove it in the next section.

Remark. To be precise, we have to remind that our cochain spaces depend on a small parameter ε. Of course, we simply take ε smaller each time it is necessary to guarantee that the final result does not depend on ε.

6 Surgery

Let V be a manifold with a convex border and $f : V \times \mathbb{R}P^N \to \mathbb{R}$ a C^1-function such that $f_v \in \mathcal{Q}$, $\forall v \in V$. The function f is regular if and only if for any $(v, \bar{x}) \in V \times \mathbb{R}P^N$ such that $x \in \ker f_v$ there exists $\xi \in T_v V$ such that $\langle \frac{\partial f}{\partial v}(v, \bar{x}), \xi \rangle > 0$.

We say that f is *strongly regular* if for any $v \in V$ such that $\ker f_v \neq 0$ there exists $\xi \in T_v V$ such that $\langle \frac{\partial f}{\partial v}(v, \bar{x}), \xi \rangle > 0$ for any $x \in \ker f_v \cap S^N$.

In other words, for the regularity to be strong we ask for ξ in the inequality to be one and the same for all $x \in \ker f_v \cap S^N$. Here is a typical example of a regular but not strongly regular map:

$$V = \{q \in \mathcal{Q} : \operatorname{tr} q = 0, \ |q| \leq 1\}, \quad f(q, \bar{x}) = q(x). \tag{5}$$

Here and below we use the following notations: $\operatorname{tr} q$ is the trace of the symmetric operator on $\mathbb{R}^{N+1}$ associated to q, $\langle q_1, q_2 \rangle$ is the trace of the product of the operators associated to q_1 and q_2, $|q| = \sqrt{\langle q, q \rangle}$. Strong regularity is violated at $q = 0$.

Lemma 3. *If $f \in \mathfrak{F}_{\mathcal{Q}}$ is in the general position, then f is strongly regular.*

Proof. Let $q \in \mathcal{Q}$ and $\ker q \neq 0$; then $q \in \Lambda^0_{j,m}$ for some j, m. It is easy to see that $T_q \Lambda^0_{j,m}$ is the kernel of the linear map $q' \mapsto q'|_{\ker q}$, $q' \in \mathcal{Q}$. Hence the transversality of the map $v' \mapsto f_{v'}$, $v' \in V$, to $\Lambda^0_{j,m}$ at $v \in V$ is equivalent to the surjectivity of the map $\xi \mapsto \left\langle \frac{\partial f}{\partial v}(v, \cdot), \xi \right\rangle\big|_{\ker f_v}$, $\xi \in T_v V$, and implies the existence of $\xi \in T_v V$ such that the quadratic form $\left\langle \frac{\partial f}{\partial v}(v, \cdot), \xi \right\rangle$ is positive definite on $\ker f_v$. $\quad \square$

Remark. We actually proved more than stated: for f to be strongly regular it is sufficient that the map $v \mapsto f_v$, $v \in M$ is transversal to submanifolds $\Lambda^0_{j,m}$; transversality to $\Lambda_{j,m}$ is not necessary.

We say that a regular homotopy f_t, $t \in [0, 1]$, is strongly regular if all f_t are strongly regular. Example: take f as in (5), $\alpha \in [0, 1)$ and the homotopy $f_t = f + t - \alpha$; then f_t is strongly regular for all t except of $t = \alpha$. We'll show later that this example is in a sense a universal model of a generic regular but not strongly regular homotopy.

Lemma 4. *Assume that $f_t \in \mathfrak{F}_{\mathcal{Q}}$, $f_t : V \times \mathbb{R}P^N \to \mathbb{R}$, $t \in [0, 1]$, is a strongly regular homotopy. Then there exists a smooth family of diffeomorphisms $F_t : V \to V$,[2], such that $F_0 = id$, $F_t(V^j_{f_0}) \subset V^j_{f_t}$, $\forall t \in [0, 1]$, $j = 1, \ldots, N+1$.*

The proof is similar to the proof of Lemma 1. It is sufficient to find a smooth vector field X_t on V such that the equality $\lambda_j(f_{t_v}) = 0$ implies:

$$\left\langle \frac{\partial f_t}{\partial v}(v, \bar{x}), X_t(v) \right\rangle > 0, \quad \forall x \in \ker f_{t_v} \cap S^N. \tag{6}$$

Indeed, fix t and v and consider a trajectory $v(\tau)$ of the flow generated by the field X_τ such that $v(t) = v$. Inequality (6) implies that for any smaller than t and sufficiently close to t number τ the quadratic form $f_{\tau v(\tau)}$ is negative definite on the linear hull of the eigenvectors of the form f_{t_v} corresponding to the eigenvalues $\lambda_j(f_{t_v}), \ldots \lambda_{N+1}(f_{t_v})$. Hence $\lambda_j(f_{\tau v(\tau)}) < 0$, according to the minimax principle for the eigenvalues of a symmetric operator. We obtain that any started in $V^j_{f_0}$ trajectory stays in $V^j_{f_t}$ for all $t \in [0, 1]$.

The existence of a desired vector field is guaranteed by the strong regularity assumption. $\quad \square$

Lemma 3 immediately implies the following:

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[2]If $\partial V \neq \emptyset$, then $F_t(V)$ may be a proper subset of V.

Corollary 2. *Strongly regular homotopies preserve the page $E^2_{i,j}$ of the spectral sequence $E^r_{i,j}$ described in Section 5.*

A routine transversality technique gives the following:

Proposition 6. *Let $\tilde{f}_t \in \mathfrak{F}_{\mathcal{Q}}$, $\tilde{f}_t : V \times \mathcal{Q} \to \mathbb{R}$, $t \in [0,1]$ be a regular homotopy and $\tilde{f}_0, \tilde{f}_1$ are in the general position. Then there exists an arbitrarily C^0-close to $\tilde{f}_t$ regular homotopy f_t such that $f_0 = \tilde{f}_0$, $f_1 = \tilde{f}_1$; the function $f_t \in \mathfrak{F}_{\mathcal{Q}}$ is not in the general position only for a finite number of values of the parameter $t \in (0,1)$, and for any f_t that is not in the general position there exists exactly one point v_t where the map $v \mapsto f_{t_v}$, $v \in V$, is not transversal to a submanifold $\Lambda_{j,m}$ or $\Lambda^0_{j,m}$ Moreover, if $v_t \in \text{int } V$, $f_{t_{v_t}} \in \Lambda^0_{j,m}$ and the map $v \mapsto f_{t_v}$, $v \in V$, is not transversal to $\Lambda^0_{j,m}$ at v_t, then the following conditions are satisfied:*

- *The image of the linear map $\frac{\partial f_t}{\partial v}(v_t, \cdot)\big|_{\ker f_{t_{v_t}}}$ from $T_{v_t}V$ into the space of quadratic forms on $\ker f_{t_{v_t}}$ is a subspace of codimension 1 in the space of quadratic forms and the orthogonal complement to this subspace is generated by $\frac{\partial}{\partial \tau}\big|_{\tau = t}(f_{\tau v_t}|_{\ker f_{t_{v_t}}})$.*

- $\frac{\partial}{\partial \tau}\big|_{\tau = t}(f_{\tau v_t}|_{\ker f_{t_{v_t}}})$ *is a nondegenerate quadratic form.*

- *The Hessian of the map $v \mapsto f_{t_v}|_{\ker f_{t_{v_t}}}$, $v \in V$ at v_t is a nondegenerate quadratic form on the kernel of the map $\frac{\partial f_t}{\partial v}(v_t, \cdot)\big|_{\ker f_{t_{v_t}}}$.*

If $v_t \in \partial V$ and the map $v \mapsto f_v$, $v \in \partial V$, is not transversal to $\Lambda^0_{j,m}$, then the same conditions are satisfied for $f_{\tau \partial V}$ in place of f_τ, and the linear map $\frac{\partial f_t}{\partial v}(v_t, \cdot)\big|_{\ker f_{t_{v_t}}}$ from $\text{span } T_{v_t}V$ into the space of quadratic forms on $\ker f_{t_{v_t}}$ is surjective.

□

We are now ready to state a local version of the homotopy invariance property.

Proposition 7. *In the setting of Proposition 6, let $t \in (0,1)$ be such that the map $v \mapsto f_{t_v}$, $v \in V$, is not in the general position. Then there exist a neighborhood O_{v_t} of v_t in V and a neighborhood o_t of t in $(0,1)$ such that the inclusions $\{\tau\} \times O_{v_t} \hookrightarrow o_t \times O_{v_t}$, $\tau \in o_t$, induce isomorphisms $\hat{H}_{\mathcal{Q}}(F_{o_t \times O_{v_t}}) \cong \hat{H}_{\mathcal{Q}}(f_{\tau O_t})$, where $F_{(\tau, v)} \doteq f_{\tau v}$.*

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General "global" homotopy invariance property easily follows from Proposition 7. Indeed, a singularity at (t, v_t) does not influence relative cohomologies for the pairs $([0,1] \times V, o_t \times O_{v_t})$, (V, O_{v_t}) and the inclusion

$$(\{\tau\} \times V, \{\tau\} \times O_{v_t}) \hookrightarrow (o_t \times V, o_t \times O_{v_t})$$

induces an isomorphism $\hat{H}_{\mathcal{Q}}(F_{o_t \times V}, F_{o_t \times O_{v_t}}) \cong \hat{H}_{\mathcal{Q}}(f_{\tau V}, f_{\tau O_t})$. The exact sequences of the pairs $(F_{o_t \times V}, F_{o_t \times O_{v_t}})$, $(f_{\tau V}, f_{\tau O_t})$ and the five lemma imply that the inclusion $\{\tau\} \times V \hookrightarrow o_t \times V$ induces an isomorphism $\hat{H}_{\mathcal{Q}}(F_{o_t \times V}) \cong \hat{H}_{\mathcal{Q}}(f_\tau)$.

Proof of Proposition 7. First assume that the map $v \mapsto f_{tv}$, $v \in V$, is transversal to all submanifolds $\Lambda^0_{j,m}$. Then f_t is strongly regular (see the Remark after Lemma 3). Hence $\tau \mapsto f_{\tau O_{v_t}}$, $\tau \in o_t$, is a strongly regular homotopy for appropriate neighborhoods O_{v_t}, o_t. Moreover, for any $\tau_0 \in o_t$ the maps $(\tau, v) \mapsto f_{\tau v}$ and $(\tau, v) \mapsto f_{\tau_0 v}$ on $o_t \times O_{v_t}$ are strongly regular homotopic. Hence $F_{o_t \times O_{v_t}}$ and $f_{\tau_0 O_{v_t}}$ have equal pages $E^2_{i,j}$.

On the other hand, $F_{o_t \times O_{v_t}}$ is regularly homotopic to a constant family $(\tau, v) \mapsto f_{t v_t} + \varepsilon$ according to the general localization result of Section 3. Moreover, this regular homotopy is strongly regular in the case under consideration and preserves the page $E^2_{i,j}$. The page $E^2_{i,j}$ of the constant family has only one nonzero column and the same is true for the families $F_{o_t \times O_{v_t}}$ and $f_{\tau_0 O_{v_t}}$. In particular, $E^2_{i,j} = E^\infty_{i,j}$ are equal fot these families.

It remains to study the case when $f_{t v_t} \in \Lambda^0_{j,m}$ and the map $v \mapsto f_{tv}$, $v \in V$, is not transversal to $\Lambda^0_{j,m}$ at v_t. Of course it is sufficient to prove the isomorphism $\hat{H}_{\mathcal{Q}}(F_{o_t \times O_{v_t}}) \cong \hat{H}_{\mathcal{Q}}(f_{\tau O_{v_\tau}})$ for one particular τ greater than t and one τ smaller than t.

We denote by Q_t the space of quadratic forms on $\ker f_{v_t}$, $Q_t = \mathcal{Q}(m-1)$. Given $q \in \mathcal{Q}$, let $E_q \subset \mathbb{R}^{N+1}$ be the linear hull of the eigenvectors of q corresponding to the eigenvalues $\lambda_j(q), \ldots, \lambda_{j+m-1}(q)$ and $\pi_q : E_q \to \ker f_{v_t}$ be the restriction to E_q of the orthogonal projector of $\mathbb{R}^{N+1}$ on $\ker f_{v_t}$. Note that $E_{f_{v_t}} = \ker f_{v_t}$ and $\pi_{f_{v_t}} = id$. We work in a small neighborhood of f_{v_t} in $\mathcal{Q}$ and may assume that E_q is transversal to the orthogonal complement of $\ker f_{v_t}$ and π_q is invertible.

Consider a map $\Phi : q \mapsto q \circ \pi_q^{-1}$ from a neighborhood of f_{v_t} to Q_t. It is a rational map and its differential at the point f_{v_t} sends a form q to $q\big|_{\ker f_{v_t}}$. Hence Φ is a submersion of a neighborhood of f_{v_t} on a neighborhood of the origin in Q_t. Moreover, $\lambda_i(\Phi(q)) = \lambda_{j+i-1}(q)$, $i = 1, \ldots, m$.

We take a sufficiently small neighborhood O_{v_t} of v_t in V and a close to t parameter $\tau \in [0, 1]$ and define $g_\tau : O_{v_t} \to \mathbb{R}$ by the formula: $g_{\tau v} = \Phi(f_{\tau v})$. Then $g_\tau \in \mathfrak{F}_{Q_t}$ and the following equalities are valid[3]:

$$V_{g_\tau}^i = V_{f_\tau}^{i+j-1} \cap O_{v_t}, \quad g_\tau^{-1}(\Lambda_{i,k}) = f_\tau^{-1}(\Lambda_{i+j-1,k}) \cap O_{v_t},$$

$i = 1, \ldots, m-1$, $k = 2, \ldots, n-i+1$. Moreover, $O_{v_t} \subset V_f^{j-1}$, $O_{v_t} \cap V_f^{j+m} = \emptyset$.

It follows that the statement of Proposition 7 for $f_\tau \in \mathfrak{F}_Q$ is equivalent to the same statement for $g_\tau \in \mathfrak{F}_{Q_\tau}$.

We have: $g_{\tau v_t} = 0$. The family $G : (\tau, v) \mapsto g_{\tau v}$, $(\tau, v) \in o_t \times O_{v_t}$ is in the general position and is strongly regular homotopic to a constant family $(\tau, v) \mapsto c$, $c > 0$, if o_t and O_{v_t} are sufficiently small. Hence $\hat{H}_{q_t}(G_{o_t \times O_{v_t}}) = 0$.

In what follows, we tacitly substitute o_t and O_{v_t} by smaller neighborhoods each time it is necessary without changing notations. First we study the case $v_t \in int\, V$ and then explain how the case $v_t \in \partial V$ is reduced to the previous one.

To go ahead we need convenient coordinates in O_{v_t}. We coordinatize O_{v_t} by the product of two balls, $O_{v_t} = U \times B = \{(u, q) : u \in U, \; q \in B\}$, where $U \subset \ker \frac{\partial g_t(v_t)}{\partial v}$, $B \subset im \frac{\partial g_t(v_t)}{\partial v}$, in such a way that $v_t = (0, 0)$ in our coordinates and

$$\frac{\partial g_t(v_t)}{\partial v} : (u, q) \mapsto q, \quad u \in \ker \frac{\partial g_t(v_t)}{\partial v}, \; q \in im \frac{\partial g_t(v_t)}{\partial v}.$$

We also set $q_0 = \frac{\partial g_\tau(v_t)}{\partial \tau}\big|_{\tau=t}$. Then B is a ball in the hyperplane $q_0^\perp \subset Q_t$. Recall that q_0 is a nondegenerate quadratic form. Moreover, we assume that the Hessian of the map $v \mapsto g_{tv}$ at v_t is normalized. This means that $\ker \frac{\partial g_t(v_t)}{\partial v} = span\, U$ is splitted in two subspaces, $span\, U = \mathbb{R}^{i_+} \oplus \mathbb{R}^{i_-}$, and

$$\frac{\partial^2 g_t(0,0)}{\partial u^2}(u) = 2(|u_+|^2 - |u_-|^2)q_0, \quad u = (u_+, u_-) \in U, \; u_\pm \in \mathbb{R}^{i_\pm}.$$

Now we apply a blow-up procedure with a small parameter $\varepsilon > 0$. We set:

$$\varphi_s^\varepsilon(u, q) = \frac{1}{\varepsilon^2} g_{t+\varepsilon^2 s}(\varepsilon u, \varepsilon^2 q), \quad |s| \leq 1, \; (u, q) \in U \times B.$$

Note that the multiplication of a quadratic form by a positive number does not change the signs and multiplicities of the eigenvalues. Hence the spectral

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[3]for simplicity, we keep symbol g_τ for the map $v \mapsto g_{\tau v}$ as in Section 5.

sequence $E_{i,j}^r$ for φ_s^ε is equal to one for $(g_\tau)_{(\varepsilon U) \times (\varepsilon^2 B)}$ with $\tau = t + \varepsilon^2 s$. We have:

$$\varphi_s^\varepsilon(u, q) = q + (|u_+|^2 - |u_-|^2 + s)q_0 + O(\varepsilon).$$

Now fix parameter $s \neq 0$. If ε is small enough (how small, depends on s), then the function φ_s^ε is homotopic to φ_s^0 in the class of functions in the general position.

What remains is to prove that $\hat{H}_{Q_t}(\varphi_s^0) = 0$. The following terminology will be useful: given $\varphi : V \to Q_t$, $\varphi \in \mathfrak{F}_{Q_t}$, and a homotopy retraction $h_\tau : V \to V$, $\tau \in [0, 1]$, we say that h_τ is monotone for φ if $V_{\varphi \circ h_\tau}^J \subset V_\varphi^j$, $j = 1, \ldots m$, $\tau \in [0, 1]$. The homotopy $\tau \mapsto \varphi \circ h_\tau$ induced by a monotone deformation retraction preserves the page $E_{i,j}^2$, d_2 of the spectral sequence.

We study separately three cases.

1. Quadratic form q_0 is sign-indefinite. In this case $q_0^\perp$ contains a positive definite form $\hat{q}$. Moreover, if s is sufficiently small then $\hat{q} + sq_0$ is a positive definite form. In this case a deformation retraction $h_\tau(u, q) = \left((1 - \tau)^{\frac{1}{2}} u, \tau \hat{q} + (1 - \tau)q \right)$ is monotone for φ_s^0. Indeed,

$$\varphi_s^0(h_\tau(u, q)) = \tau(\hat{q} + sq_0) + (1 - \tau)\left(q + (|u_+|^2 - |u_-|^2 + s)q_0 \right). \quad (7)$$

The signature of a quadratic form (i. e. the numbers of positive and negative eigenvalues) does not change under a linear change of coordinates in $\mathbb{R}^m$, although the eigenvalues do change. Take coordinates such that the form $\hat{q} + sq_0$ is represented by a scalar matrix. In these coordinates, eigenvalues of the form (7) are linear functions of τ. We have: $\varphi_s^0(h_1(u, q)) \equiv \hat{q} + sq_0$. Hence $E_{i,j}^2 = 0$.

2. Quadratic form sq_0 is positive definite. Then the deformation retraction $h_\tau(u, q) = \left((1 - \tau)^{\frac{1}{2}} u, (1 - \tau)q \right)$ is monotone for φ_s^0 and $\varphi_s^0(h_1(u, q)) \equiv sq_0$. Hence $E_{i,j}^2 = 0$.

3. Quadratic form sq_0 is negative definite. In this case, the page $E_{i,j}^2$ is very far from being zero. We already mentioned that the transformation of Q_t induced by a linear change of coordinates in $\mathbb{R}^m$ does not change the signs of eigenvalues and thus the groups $E_{i,j}^2$ of the spectral sequences associated to elements of $\mathfrak{F}_{Q_t}$. It is important that the differentials d_2 do not change as well. The last statement needs a justification since the submanifolds $\Lambda_{j,2} \subset Q_t$ do depend on the choice of coordinates in $\mathbb{R}^m$. The differential d_2 of the

spectral sequence $E_{i,j}^r$ does not depend on the choice of coordinates because it is equal to the differential d_2 of the constructed in [2] spectral sequence $F_{i,j}^r$ (see Section 5), and $F_{i,j}^r$ is the Leray spectral sequence of a map that respects changes of coordinates.

Now take coordinates in $\mathbb{R}^m$ such that the form q_0 is represented by a scalar matrix. Then B is a ball in the space of symmetric matrices with zero trace. If $q_0 > 0$, then the deformation retraction $(u_+, u_-, q) \mapsto (u_+, (1-\tau)u_-, q)$, $\tau \in [0,1]$, is monotone for φ_s^0. Similarly, if $q_0 < 0$, then the deformation retraction $(u_+, u_-, q) \mapsto ((1-\tau)u_+, u_-, q)$, $\tau \in [0,1]$, is monotone.

Next lemma completes the proof of Proposition 7 in the case $v_t \in int\, V$,

Lemma 5. *Let* $0 < s < 1$,

$$U = \{u \in \mathbb{R}^k : |u|^2 \leq 2\}, \quad \mathbb{B} = \{q \in \mathcal{Q} : \operatorname{tr} q = 0,\ \|q\| \leq 1\},$$

and the map $\varphi : U \times B \to \mathcal{Q}$, $\varphi \in \mathfrak{F}_{\mathcal{Q}}$, *is defined by the formula:* $\varphi(u,q) = q + |u|^2 - s$. *Then the page* $E_{i,j}^3$ *of the associated to* φ *spectral sequence* $E_{i,j}^r$ *is zero.*

Proof. We have to prove that the cochain complex (E^2, d_2) is exact. It is not at all obvious but it is actually proved in [1, Th. 2]. Indeed, let us show that the complex (E^2, d_2) can be naturally identified with complex (1) from [1], where $n = N + 1$.

We set: $M^j = \{q \in \mathbb{B} : \|q\| = 1,\ \lambda_{N-j+1}(q) \neq \lambda_{N+1}(q)\}$, like in [1] (note that the eigenvalues have the reversed ordering in [1]). Recall that $E_{i,j}^2 = H^i(V, V_\varphi^{j+1})$, where $V = U \times \mathbb{B}$. A simple homotopy that moves only eigenvalues of symmetric matrices keeping fixed the eigenvectors gives a homotopy equivalence of pairs:

$$\left(U \times \mathbb{B}, V_\varphi^{j+1}\right) \cong \left(U \times \mathbb{B}, (U \times M^{N-j}) \cup (\partial U \times \mathbb{B})\right).$$

Hence $E_{i,j}^2 = H^{i-k}(\mathbb{B}, M^{N-j})$; moreover, natural isomorphism of $E_{\cdot,\cdot}^2$ and $H^{\cdot-k}(\mathbb{B}, M^{N-\cdot})$ transforms d_2 in the differential of the exact complex (1) from [1]. $\square$

Let $v_t \in \partial V$; we consider the maps $g_\tau|_{\partial V}$, take appropriate coordinates, and apply the blow-up procedure as we did for g_τ in the case of an interior point v_t. We arrive to the map $\varphi_s^0 : (u,q) \mapsto q + (|u_+|^2 - |u_-|^2 + s)q_0$ extended to $U \times B^+$ or $U \times B^-$, where $B^\pm$ is the intersection of a ball in Q_t with the

half-space $\{q \in Q_t : \pm\langle q, sq_0\rangle \geq 0\}$. We denote these extensions by $\varphi_s^{\pm}$. What remains is to prove that $\hat{H}_{Q_t}(\varphi_s^{\pm}) = 0$.

If sq_0 is not negative definite and $|s|$ is sufficiently small, then simple monotone deformation retractions transform $\varphi_s^{\pm}$ into a positive constant. The same is true for φ_s^{+} with a negative definite sq_0. The only remaining possibility is φ_s^{-} with a negative definite sq_0. In this case, a deformation retraction $h_\tau(u, q) = \left(u, q - \tau\frac{\langle q, q_0\rangle}{|q_0|^2}q_0\right)$, $\tau \in [0, 1]$, is monotone and transforms φ_s^{-} in the already studied φ_s^{0} defined on $U \times B$. $\qquad \square$

Remark. We have shown that local disturbance in the page E^2 provoked by a violation of the strong regularity during a regular homotopy is totally calmed in the page E^3. However, this fact does not imply regular homotopy invariance of E^3 because the complexes E^2, d_2 do not satisfy the exact sequence "axiom" and invariance of their local cohomologies does not imply invariance of the global ones.

7 Example

Let $\mathbb{H}$ be the quaternion algebra, $\mathbb{H} = \mathbb{R} \oplus \mathbb{R}^3$, where $\mathbb{R}$ is the real line and $\mathbb{R}^3$ is the space of purely imaginary quaternions, $\mathbb{R}^3 = \{x \in \mathbb{H} : \bar{x} = -x\}$. We take $a \in \mathbb{R}^3 \setminus \{0\}$ and consider a quadratic map $\varphi : \mathbb{H} \to \mathbb{R}^3$ defined by the formula $\varphi(x) = \bar{x}ax$. Then $|\varphi(x)| = |a||x|^2$. In particular, $\varphi^{-1}(x) = 0$. The restriction of φ to S^3 is just adjoint representation of the group $\mathrm{SU}(2) = S^3$ and a realization of the Hopf bundle $S^3 \to S^2$. Now consider a family of quadratic forms $\varphi_p^* \in \mathcal{Q}(3)$, $p \in B^3 = \{p \in \mathbb{R}^3 : |p| \leq 1\}$, where $\varphi_p^*(x) = \langle p, \varphi(x)\rangle$; then $\varphi^* \in \mathfrak{F}_{\mathcal{Q}(3)}$, $\hat{H}_{\mathcal{Q}(3)}(\varphi^*) = 0$.

We have $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C} = \mathbb{C}^2$. Quadratic forms φ_p^* are thus real quadratic forms on $\mathbb{C}^2$. It is easy to see that they are Hermitian quadratic forms whose Hermitian matrices have zero traces. In other words, $span\{\varphi_p^* : p \in B^3\} = i\mathrm{su}(2)$. Eigenspaces of the associated to φ_p^* symmetric operators in $\mathbb{R}^4$ are complex lines; hence the eigenvalues are double and we have

$$\lambda_1(\varphi_p^*) = \lambda_2(\varphi_p^*) = -\lambda_3(\varphi_p^*) = -\lambda_4(\varphi_p^*),$$

$$V_{\varphi^*}^1 = V_{\varphi^*}^2 = B^3 \setminus \{0\}, \quad V_{\varphi^*}^3 = V_{\varphi^*}^4 = \emptyset.$$

Let ς be a small quadratic form, then $\phi^* - \varsigma$ is regularly homotopic to φ^* and $\hat{H}_{\mathcal{Q}(3)}(\varphi^* - \varsigma) = 0$. Moreover, $\varphi^* - \varsigma$ is in the general position for almost every ς.

Assume that ς is positive definite; then $V^1_{\varphi^*-\varsigma}$, $V^2_{\varphi^*-\varsigma}$ are complements to (small) contractible neighborhoods of 0, $V^3_{\varphi^*-\varsigma} = V^4_{\varphi^*-\varsigma} = \emptyset$. Indeed, the number of positive eigenvalues of the operator associated to a quadratic form does not depend on the choice of the Euclidean structure. If we choose a form $\frac{1}{\varepsilon}\varsigma$ as the Euclidean structure, then $\lambda_i(\varphi^*_p - \varsigma) = \lambda_i(\varphi^*_p) - \varepsilon$.

The page E^2 of the spectral sequence E^r for $\varphi^* - \varsigma$ has a form:

$\mathbb{Z}_2$	0	0	0
$\mathbb{Z}_2$	0	0	0
0	0	0	$\mathbb{Z}_2$
0	0	0	$\mathbb{Z}_2$

Hence the differentials $d_3 : E^2_{0,j+1} \to E^2_{3,j-1}$, $j = 2, 3$, are not zero. We are in the situation described in Section 5 (see the paragraph with formula ($*$) and the next paragraph). It follows that the linking number mod 2 of $(\varphi^* - \varsigma)^{-1}(\Lambda_{2,2})$ with $(\varphi^* - \varsigma)^{-1}(\Lambda_{1,2})$ and with $(\varphi^* - \varsigma)^{-1}(\Lambda_{3,2})$ are nonzero.

The Proposition stated in the Introduction can be easily derived from this fact. We start from the case of generic S_0. First of all, $C^{S_0+tI}_i = C^{S_0}_i$ for any scalar matrix tI. Hence we may assume that S_0 is the matrix of a negative definite quadratic form. It is sufficient to compute linking numbers of $C^{S_0}_2$ with $C^{S_0}_1$ and with $C^{S_0}_3$ in a very big ball $\frac{1}{\varepsilon}B^3$. The multiplication by ε transforms $C^{S_0}_j$ into $C^{\varepsilon S_0}_j = (\varphi^* - \varsigma)^{-1}(\Lambda_{j,2})$, $j = 1, 2, 3$, where ς is the quadratic form represented by the matrix $-\varepsilon S_0$.

We have proved the statement about linking numbers in the case of generic S_0. Now take any S_0 and present it as the limit of a sequence of generic ones, $S_0 = \lim\limits_{n\to\infty} S^n_0$. Any limiting point of the sequence of sets $C^{S^n_0}_j$ as $n \to \infty$ belongs to $C^{S_0}_j$. The curves $C^{S^n_0}_2$ are uniformly bounded, hence $C^{S_0}_2 \neq \emptyset$. The curves $C^{S^n_0}_1$ and $C^{S^n_0}_3$ are linked with $C^{S^n_0}_2$ and cannot escape to infinity; hence $C^{S_0}_1$ and $C^{S_0}_3$ are also nonempty.

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