On the local structure of optimal trajectories in $\mathbf{R}^3$

Andrei A. Agrachev [1] Mario Sigalotti [2]

41/2002/M

[1]SISSA-ISAS, via Beirut 2-4, 34014 - Trieste, Italy and Steklov Mathematical Institute, ul. Gubkina 8, Moscow, Russia; E-mail: agrachev@sissa.it.

[2]SISSA-ISAS, via Beirut 2-4, 34014 - Trieste, Italy; E-mail: sigalott@sissa.it

Abstract

We analyze the structure of a control function $u(t)$ corresponding to an optimal trajectory for the system $\dot{q} = f(q) + u\, g(q)$ in a three-dimensional manifold, nearby a point where some nondegeneracy conditions are satisfied. The kind of optimality which is studied includes time-optimality. The control turns out to be the concatenation of some bang and some singular arcs. Studying the index of the second variation of the switching times, the number of such arcs is bounded by four.

Key Words: optimal control, Lie brackets.

AMS subject classification: 49K15, 49K30.

1 Introduction

Consider the time-optimal control problem

$$\dot{q} = f(q) + u\, g(q), \quad u \in [-1,1]\,, \tag{1}$$

on a 3-dimensional manifold M, where f and g are two smooth vector fields on M and the admissible controls are all measurable functions $u : t \mapsto u(t) \in [-1,1]$. The kind of results we are interested in are local regularity properties for the control function corresponding to optimal trajectories. Namely: we fix $q_0 \in M$ and we study whether *locally at* q_0 the control function corresponding to a time-optimal trajectory is piecewise smooth, meaning by this that it exists a neighborhood U of q_0 and a time $T > 0$ such that the control function corresponding to any time-optimal trajectory of the system (1) contained in U and defined on a time-interval of length less or equal than T is piecewise smooth. We are interested in giving an upper bound to the number of smooth pieces and in describing the possible concatenations (for instance we want to know how many pieces are bang, that is: the control restricted to them is the constant function $+1$ or -1).

As it is well known (see [14]), any irregularity of the optimal control is possible, if no further *regularity* assumption is made. The language in which this regularity will be expressed is the one of "nonresonance conditions" on the configuration of iterated Lie brackets between f and g evaluated at the fixed point q_0. This is natural since Lie bracket relations form a complete system of differential invariants of the pair of vector fields f, g.

The first step in this direction is to study the structure of the trajectories near the point where some prescribed triples of iterated Lie brackets are linearly independent. If a local regularity property is proved under such conditions, by standard transversality considerations it follows that: for a generic pair of vector fields f and g, for a generic point $q_0 \in M$ the regularity property holds locally at q_0. The first result of this kind appeared in [13] and asserts that if each of the tensor fields $f \wedge g \wedge [f,g]$, $g \wedge [f,g] \wedge [f+g,[f,g]]$ and $g \wedge [f,g] \wedge [f-g,[f,g]]$ does not vanish at q_0, then locally at q_0 any time-optimal bang-bang trajectory has at most two switchings.

A satisfactory understanding of the three-dimensional problem would be for us to individuate a regularity property which holds locally at every point of M for a generic pair of vector fields (f,g), in analogy with what has already been done for two-dimensional manifolds by Sussmann in [12]. By Thom's Transversality Theorem we know that this reduces to study all Lie brackets configurations with up to three nontrivial independent relations of linear dependency between its elements. Up to now, in addition to the cited result by Sussmann, there have been proved similar upper bounds in other situations: by Bressan [5] in the case of trajectories steering to an equilibrium point for f under some extra nondependency conditions; by Schättler [9] for bang-bang trajectories for a generic pair (f, g), locally at all points in which $f \wedge g \wedge [f, g] \neq 0$ (for trajectories which are not apriori bang-bang Schättler [10] requires that some extra nondependency conditions hold at the point). In [3] it is stated the following complementary result, which generalizes the one in [5]: if $g \wedge [f, g] \wedge [f \pm g, [f, g]](q_0) \neq 0$, then locally at q_0 a bang-bang time-optimal trajectory has no more than two switches. The present paper furnish a detailed proof of that statement. The same result had also been proved by Sussmann [11] and Krener and Schättler [15] by different methods.

In addition, the present paper gives a bound for all Lie brackets configurations with up to one dependency relations for trajectories with no apriori restrictions, and also discusses in which sense the result is sharp. The main results are contained in Theorem 3 in Section 3 and in Lemma 2 in Subsection 4.3 and can be summarized as follows, in terms of generic properties for (f, g) and of properties holding at any point of M, in the local sense introduced above.

Theorem 1 *Let $V_2^4(M)$ be the space of all pairs of $\mathcal{C}^4$ vector fields on M, endowed with the Whitney topology. For any element (f, g) from an open everywhere dense subset of $V_2^4(M)$ there exist $W_1 \subset W_2 \subset M$, where W_1 and W_2 are, respectively, a one-dimensional and a two-dimensional stratified set in M, such that:*

- *for any point q_0 in $M \setminus W_2$, locally at q_0 any time-optimal trajectory of the system (1) is a finite concatenation of at most three between bang and singular arcs;*
- *for any point q_0 in $W_2 \setminus W_1$, locally at q_0 any time-optimal trajectory of the system (1) is a finite concatenations of at most four between bang and singular arcs.*

Proposition 1 *For any (f, g) from an open subset of $V_2^4(M)$ there exists a 2-dimensional submanifold $U_2 \subset M$ such that for any $q_0 \in U_2$ and any $T > 0$ it exists a starting from q_0 bang-bang trajectory of time-length less than T with three switches, which is locally (in the $\mathcal{C}^0$ topology) time-optimal.*

The motivations for the study of this topic are mainly two: (i) to give apriori restrictions on the local structure of optimal trajectories is a crucial step in the direction of the description of the local optimal synthesis and (ii) there is a sincere mathematical interest in understanding the generic behavior of optimal controls. Fuller [6] first proposed a polynomial system of the kind studied here such that the switching time moments of the optimal control form a convergent sequence (we call *chattering* a control of this kind). Since there it has been an important question to understand whether this phenomenon is structurally stable (i.e. cannot be eliminated by a small perturbation of the system) or not. In big enough dimensions it most likely is stable: stable chattering extremals were constructed in [8, 16], though the optimality of these extremals is not proved. Our paper is one more step in the demarcation of the non-chattering territory, at least in dimension 3. Another problem is whether Fuller phenomenon is the worst possible stable behavior. Lemma 3 in Section 5 gives a partial answer to this question, stating that under some very weak conditions (which hold everywhere for a generic pair (f, g) in any dimension) an optimal control with values in $\{-1, 1\}$ either has a finite number of switches or is such that its restriction to a subinterval is chattering.

2 A Second Order Optimality Condition

Let M be a smooth manifold and f, g two smooth vector fields on M. A *trajectory* of the system

$$\dot{q} = f(q) + ug(q), \quad u \in [-1, 1], \tag{2}$$

is an absolutely continuous curve $t \mapsto q(t) \in M$ for which it exists a measurable *control function* $t \mapsto u(t) \in [-1, 1]$ such that (2) is verified for almost every t in the domain of $q(\cdot)$. For all $T > 0$ and $q_0 \in M$ we define

$$A(T, q_0) = \{q(T) \mid q : [0, T] \to M \text{ is a trajectory of (2) such that } q(0) = q_0\}.$$

Let $q : [0, T] \to M$ be a trajectory of (2) and u the corresponding control function. By the Pontryagin Maximum Principle we know that if $q(T) \in \partial A(T, q(0))$ then q is *extremal*, that is: there exist $c \in \mathbf{R}$ and an absolutely continuous covector trajectory $p : [0, T] \to T^*M$ such that $p(t) \in T^*_{q(t)}M \setminus \{0\}$ for every $t \in [0, T]$ and which verifies for almost every t the equation

$$\dot{p}(t) = -p(t)(Df(x(t)) + u(t)Dg(x(t))), \tag{3}$$

and the relation

$$\langle p(t), (f + u(t)g)(q(t))\rangle = \min_{v \in [-1,1]} \langle p(t), (f + vg)(q(t))\rangle \equiv c. \tag{4}$$

We say that $q : [0, T] \to M$ is *bang-bang* if the control function $u(\cdot)$ takes values in $\{-1, 1\}$ and it exists a finite number of *switching times* $0 < t_1 < t_2 < \ldots < t_{k-1} < t_k < T$, splitting $[0, T]$ in intervals on which u is alternatively the constant $+1$ and -1. In general we say that a piece of trajectory defined on a time subinterval is a bang arc if the corresponding control is constantly equal to $+1$ or -1 (we speak, respectively, of a $+$ arc or a $-$ arc), whereas it is a singular arc if it is not a bang arc and the corresponding control is smooth. We will describe the structure of a trajectory by the standard agreement that, for instance, a $+-$S trajectory is a concatenation of a $+$, a $-$ and a singular arc.

Since we are interested in local results it is justified and convenient for us to assume that all the vector fields involved are complete. Given a complete smooth vector field h on M and a time $t \in \mathbf{R}$, we can associate the flow of h at a time t, which we will denote by e^{th}. Both h and e^{th} have a natural interpretation as operators on $\mathcal{C}^\infty(M)$: given a smooth function a on M and a point $q \in M$, $ha(q)$ is defined as the derivative of a in the direction $h(q)$ at the point q, whereas $(e^{th}a)(q) = a(e^{th}(q))$. Given two smooth vector fields h_1 and h_2 it is always possible to define their commutator (Lie bracket) according to the formula $((\mathrm{ad}h_1)h_2)a = [h_1, h_2]a = h_1(h_2 a) - h_2(h_1 a)$. In operator terms the action $e_*^{-th_1}$ of the diffeomorphism e^{-th_1} on the vector fields has the form: $e_*^{-th}h_2 = e^{th_1} \circ h_2 \circ e^{-th_1}$. The formula

$$\frac{\mathrm{d}}{\mathrm{d}t} e^{th_1} \circ h_2 \circ e^{-th_1}(q) = [h_1, e^{th_1} \circ h_2 \circ e^{-th_1}](q)\,,$$

justifies the notation $e^{t\mathrm{ad}h_1}h_2 = e^{th_1} \circ h_2 \circ e^{-th_1}$.

In the case of vector fields which are not smooth but just $\mathcal{C}^k$ the above definitions extend to the case where they are still licit, and in particular if h is $\mathcal{C}^k$ then $e^{t\mathrm{ad}h}$ is a well defined transformation of $\mathcal{C}^k$ vector fields and $k+1$ iterated Lie brackets between $\mathcal{C}^k$ vector fields are well defined $\mathcal{C}^0$ vector fields.

In order to formulate the second order optimality condition it is useful to remark the following fact. Let $q(\cdot)$ be an extremal trajectory of (2) and $p(\cdot)$ an associated covector trajectory. Fix a vector field h and two time instants t and τ in the same bang interval of $q(\cdot)$ on which the control is equal to ν. By equation (3) we have:

$$\langle p(t), h(q(t))\rangle = \langle p(\tau), e^{(t-\tau)\mathrm{ad}(f+\nu g)} h(q(t))\rangle\,. \tag{5}$$

Theorem 2 *Let $q : [0,T] \to M$, $T > 0$, be a bang-bang trajectory of (2) and let $u(t)$ be the corresponding control function with k switching times $0 < \tau_1 < \tau_2 < \ldots < \tau_k < T$. Denote by ν the value of u in $(0, \tau_1)$. Let $q(\cdot)$ be extremal and $p(\cdot)$ a corresponding covector trajectory. Assume that $p(\cdot)$ is uniquely defined (up to multiplication by a positive scalar) by (3) and (4). Let $p_0 = p(0)$ and*

$$\begin{aligned} h_0 &= f + \nu g\,, \\ h_i &= e^{\tau_1 \mathrm{ad}(f+\nu g)} e^{(\tau_2-\tau_1)\mathrm{ad}(f-\nu g)} \circ e^{(\tau_3-\tau_2)\mathrm{ad}(f+\nu g)} \circ \ldots \\ &\quad \circ e^{(\tau_i-\tau_{i-1})\mathrm{ad}(f-(-1)^i \nu g)} (f + (-1)^i \nu g) \quad i = 1, \ldots, k\,. \end{aligned}$$

Take the quadratic form

$$Q(\alpha) = \sum_{0 \le i < j \le k} \alpha_i \alpha_j \langle p_0, [h_i, h_j](q(0))\rangle\,, \tag{6}$$

defined on the space

$$\left\{ \alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbf{R}^{k+1} \;\middle|\; \sum_{i=0}^k \alpha_i = 0,\ \sum_{i=0}^k \alpha_i h_i(q(0)) = 0 \right\}. \tag{7}$$

If Q is not nonnegative defined then $q(T) \in \mathrm{int}A(T, q(0))$. Moreover, for every system (f', g') which is close enough to (f, g) in the $\mathcal{C}^1$ topology, the point $q(T)$ is in the interior of the corresponding attainable set $A_{(f',g')}(T, q(0))$.

The condition that Q is not nonnegative defined is usually rephrased by saying that the index of Q, that is: the dimension of the maximal subspace on which Q is negative defined, is strictly positive. This theorem, in a much more general setting, has been proved in [3]. We will give here just a brief sketch of the proof. Let $q_0 = q(0)$ and let $F : w(\cdot) \mapsto F(w)$ be the endpoint mapping at time T for the system

$$\begin{cases} \dot{x} = f(x) + wg(x)\,, \\ x(0) = q_0\,, \end{cases}$$

which is defined in a L^1 neighborhood of the reference control $u(\cdot)$. Take

$$G(v) = e^{-t_1(f+\nu g)} \circ e^{(t_1-t_2)(f-\nu g)} \circ \ldots \circ e^{(t_k-T)(f+(-1)^k\nu g)} F(u+v)\,,$$

which is just F, simply pullbacked by the flow generated by the control $u(\cdot)$, in order to have $G(0) = q_0$. Our aim is to prove that G is locally open at 0 and moreover that this property is stable with respect to $\mathcal{C}^1$ perturbations of our system. Let

$$\begin{aligned} h(t) &= e^{\tau_1 \operatorname{ad}(f+\nu g)} e^{(\tau_2-\tau_1)\operatorname{ad}(f-\nu g)} \circ e^{(\tau_3-\tau_2)\operatorname{ad}(f+\nu g)} \circ \ldots \\ &\quad \circ e^{(t-\tau_{i-1})\operatorname{ad}(f-(-1)^i\nu g)}(f+(-1)^i\nu g) \end{aligned}$$

,

for all $t \in [\tau_{i-1}, \tau_i]$ and all $i = 1, \ldots, k+1$, with the agreement that $\tau_0 = 0$ and $\tau_{k+1} = T$. The extremality condition (4) implies that $\langle p_0, h(t)(q_0)\rangle \geq 0$ for all $t \in [0,T]$, as we immediately deduce from (5). The uniqueness of $p(\cdot)$ means, moreover, that the closed convex cone generated by $\{h(t)(q_0) \mid t \in [0,T]\}$ is an half-space of $T_{q_0}M$, which we denote by H.

For any $\alpha = (\alpha_0, \ldots, \alpha_k) \in \mathbf{R}^{k+1}$ with $\sum_{i=0}^k \alpha_i = 0$ and for $s > 0$ small enough, we can define $w_s^\alpha(\cdot)$ as the bang-bang control with switching times $0 < t_1 + s\alpha_0 < t_2 + s(\alpha_0 + \alpha_1) < \ldots < t_k + s\sum_{i=0}^{k-1}\alpha_i < T$. Let $v_s^\alpha = u - w_s^\alpha$. We have that

$$\frac{\mathrm{d}}{\mathrm{d}s} G(v_s^\alpha)\Big|_{s=0+} = \sum_{i=0}^k \alpha_i h_i(q_0)\,,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}s^2} G(v_s^\alpha)\Big|_{s=0+} = \sum_{i=0}^k \sum_{j=i+1}^k \alpha_i\alpha_j [h_i, h_j](q_0)\,.$$

The positiveness of Q can thus be read as follows: the space defined by (7) consists of all variations of the switching times that preserve the total time T and which produce a zero first order variation in the endpoint. If a variation α exists such that $Q(\alpha)$ is negative, then the convex cone generated by $v_0 = \frac{\mathrm{d}^2}{\mathrm{d}s^2} G(v_s^\alpha)\Big|_{s=0+}$ and H is the whole $T_{q_0}M$.

Fix such an α and take $t_1, t_2, t_3 \in]0, T[\setminus \{\tau_1, \ldots, \tau_k\}$ such that the convex cone generated by v_0 and $v_i = h(t_i)(q_0)$, $i = 1, 2, 3$, is equal to $T_{q_0}M$. Define the admissible control

$$w_{(s_0,s_1,s_2,s_3)}(t)=\begin{cases} -u(t) & \text{if } t\in\cup_{i=1}^3[t_i,t_i+s_i]\,,\\ w^\alpha_{s_0}(t) & \text{otherwise}\,,\end{cases}$$

for $s_0,s_1,s_2,s_3>0$ small enough and let $v_{(s_0,s_1,s_2,s_3)}=u-w_{(s_0,s_1,s_2,s_3)}$. Fix $a_0,a_1,a_2,a_3>0$ and remark that

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}G(v_{(a_0\varepsilon,a_1\varepsilon^2/2,a_2\varepsilon^2/2,a_3\varepsilon^2/2)})\Big|_{\varepsilon=0+}=0\,,$$

and

$$\frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2}G(v_{(a_0\varepsilon,a_1\varepsilon^2/2,a_2\varepsilon^2/2,a_3\varepsilon^2/2)})\Big|_{\varepsilon=0+}=\sum_{i=0}^3 a_i v_i\,.$$

Now a standard application of the Brower fixed point theorem implies that for any continuous mapping $\mathcal{C}^0$ close to $(s_0,s_1,s_2,s_3)\mapsto G(v_{(\sqrt{s_0},s_1,s_2,s_3)})$ the image of any neighborhood of 0 in $[0,+\infty)^4$ contains a neighborhood of q_0 in M. This concludes the sketch of the proof of Theorem 2.

The theorem above naturally suggests the following definition, which makes sense for any smooth control system: we say that a trajectory $q:[0,T]\to M$ of $\dot q=f(q,u)$, where f is $\mathcal{C}^1$ with respect to both the state q and the control u, is *essential* if $q(T)$ belongs to the interior of the attainable set from $q(0)$ at time T for any $\mathcal{C}^1$-close system. In particular, if we consider time-rescaled systems (which, up to some technical use of cut-off functions, can be seen as $\mathcal{C}^1$-close to the original one, if the scaling factor is close to 1), we get that if $q:[0,T]\to M$ is essential, then $q(T)\in A(q(0),t)$ for t close to T. In particular q is not the fastest trajectory connecting $q(0)$ to $q(T)$ (i.e. it is not time-optimal) neither the slowest. This approach therefore allows a unified study of a wide range of phenomena, including time-optimality. The previous considerations justify us to call *quasi optimal* a trajectory which is not essential.

A property that quasi optimality shares with time-optimality (and in general with any optimality defined by an integral cost) is the fact that the time-reversed of a quasi optimal trajectory is quasi optimal for the time-reversed system. Indeed if $q:[0,T]\to M$ is essential for the control system $\dot q=f(q,u)$ then there exist $\delta>0$ and a neighborhood U of $q(0)$ such that for any f' δ-close to f and for any $q\in U$, $q(T)$ belongs to the attainable set from q at time T of the system $\dot q=f'(q,u)$, as we can derive by a reparameterization argument similar to the one above. Thus $q(0)$ belongs to the interior of the attainable set from $q(T)$ at time T for any system δ-close to $\dot q=-f(q,u)$.

3 The results

In this section we explicitly furnish a partition of all Lie brackets configuration with zero or one relations of linear dependency between its elements and we state the corresponding bound on the number of switches that we will prove in the next sections. For the seek of concision we set:

$$\begin{aligned} X_{\pm} &= [f \pm g, [f,g]]\,, \\ X_{\pm\mp} &= [f \pm g, [f \mp g, [f,g]]]\,, \end{aligned}$$

and in general, if w is a word with letters in $\{+,-\}$, we set $X_{\pm w} = [f \pm g, X_w]$. The two classes of Lie brackets configurations we will consider are characterized as follows:

Case 0
$g \wedge [f,g] \wedge X_+ \neq 0$
$g \wedge [f,g] \wedge X_- \neq 0$

Case 1
$g \wedge [f,g] \wedge X_+ = 0$
$g \wedge [f,g] \wedge X_{++} \neq 0$
$g \wedge [f,g] \wedge X_- \neq 0$

Every set of equalities and inequalities has to be interpreted as evaluated at q_0. We say that a Lie brackets configuration is of type 0 or 1 to mean that it satisfies the conditions of, respectively, Case 0 or 1. This partition is complete since, if in (1) we substitute g by $-g$, we obtain exactly the same system (due to the symmetricity of the control set), but the roles of $+$ and $-$ are transposed.

Our main result is the following.

Theorem 3 *Let q_0 be a point of M whose Lie brackets configuration is of the type 0 or 1. Then there exist a neighborhood U of q_0 and a time $T > 0$ for which a quasi optimal trajectory of the system (1) contained in U and of time-length less or equal than T is a finite concatenations of bang and singular arcs. In particular for T and U small enough: if we are in the conditions of Case 0 a quasi optimal trajectory is a concatenation of at most three bang arcs or of a bang a singular and a bang arc; if we are in the conditions of Case 1 a quasi optimal trajectory is a concatenation of at most four bang arcs or of two bang a singular and a bang arc and the only possible maximal concatenations including singular arcs are of the type $-{+}S\pm$ or $\pm S{+}-$.*

In the following we denote by $q(t)$ an extremal trajectory and by $p(t)$ an associated covector trajectory. Let $\varphi(t)$ be the *switching function*, that is

$$\varphi(t) = \langle p(t), g(q(t))\rangle.$$

Notice that condition (4) implies that φ assumes the value zero at the switching times of $q(\cdot)$. From Lemma 1 in [1] we know that, given a smooth vector field X, for almost every t in the domain of $q(\cdot)$, we have

$$\frac{d}{dt}\langle p(t), X(q(t))\rangle = \langle p(t), [f + u(t)g, X](q(t))\rangle\,. \tag{8}$$

In particular for almost every t we have $\dot\varphi(t) = \langle p(t), [f,g](q(t))\rangle$. This equality holds, moreover, for every t since φ is absolutely continuous and $t \mapsto$

$\langle p(t), [f,g](q(t))\rangle$ is (absolutely) continuous. Therefore φ is a $\mathcal{C}^1$ function, its derivative is absolutely continuous and

$$\ddot{\varphi}(t) = \langle p(t), [f + u(t)g, [f,g]](q(t))\rangle$$

almost everywhere. We stress that the above considerations do not depend on the dimension of M.

A technical result based on the above remarks is the following.

Lemma 1 *Let X_i, $i = 1,2,3$, be three smooth vector fields on M linearly independent at q_0. Then it exists a neighborhood U of q_0 and it exists $T > 0$ such that: for each interval I of time-length less or equal than T, for each extremal trajectory $q : I \to U$ and for each corresponding covector trajectory $p(\cdot)$, if both $x_1(t)$ and $x_2(t)$ have at least a zero in I then the sign of $x_3(t)$ is constant on I, where $x_i(t) := \langle p(t), X_i(q(t))\rangle$, $i = 1,2,3$.*

Proof. Fix a precompact neighborhood U of q_0 on which $X_1 \wedge X_2 \wedge X_3 \neq 0$. By normalizing $p(t_0)$ at the middle point of I, fixing a small T and using Gronwall inequality applied to (3) we can assume that $c \leq |p(t)| \leq C$ for every $t \in I$, for fixed $0 < c < C$. From the expressions of $\dot{x}_i$ given in (8) we easily deduce that it exists a constant K for which

$$|x_i(t)| \leq TCK\,,\ i = 1,2\,,$$

for every $t \in I$. Then $p(t)$ belongs to the compact set

$$\{p \in T^*_{q(t)}M \mid c \leq |p| \leq C,\ |x_i(t)| \leq TCK\,,\ i = 1,2\}\,,$$

which depends continuously on t. Taking a small enough T the lemma is proved. $\square$

4 The bang-bang case

In this section we aim to prove Theorem 3 for trajectories which are already known to be bang-bang. Let $q(\cdot)$ be extremal and bang-bang. Assume that $0, t_1, t_1+t_2$ are the switching times of q, with $t_i > 0$ for every i, and that $u = +1$ on $(0,t_1)$. This means that $q : [-\eta, t_1+t_2+\eta] \to M$ for some $\eta > 0$ and that u takes value $+1$ on $(0,t_1) \cup (t_1+t_2, t_1+t_2+\eta)$ and -1 on $(-\eta,0) \cup (t_1, t_1+t_2)$. Since a switching time is a zero of φ, we have:

$$\begin{aligned}
0 &= \langle p_0, g(\bar{q})\rangle, && (9)\\
0 &= \langle p_0, e^{t_1 \mathrm{ad}(f+g)} g(\bar{q})\rangle, && (10)\\
0 &= \langle p_0, e^{t_1 \mathrm{ad}(f+g)} e^{t_2 \mathrm{ad}(f-g)} g(\bar{q})\rangle, && (11)
\end{aligned}$$

where $p_0 = p(0)$ and $\bar{q} = q(0) \in U$. We show in the following the uniqueness of the covector p_0, at least for small T and U. This will allow us to apply Theorem

2 and to formulate necessary conditions for the quasi optimality of the studied trajectory in terms of the index of

$$Q(\alpha) = \sum_{0\le i<j\le 3} \alpha_i\alpha_j\langle p_0, [h_i, h_j](\bar{q})\rangle,$$

defined for all $\alpha = (\alpha_0, \alpha_1, \alpha_2, \alpha_3) \in \mathbf{R}^4$ such that

$$\sum_{i=0}^{3} \alpha_i h_i(\bar{q}) \quad = \quad 0\,, \tag{12}$$

$$\sum_{i=0}^{3} \alpha_i \quad = \quad 0\,. \tag{13}$$

Recall that

$$\begin{aligned} h_0 &= f - g\,, \\ h_1 &= f + g\,, \\ h_2 &= e^{t_1 \mathrm{ad}(f+g)}(f-g)\,, \\ h_3 &= e^{t_1 \mathrm{ad}(f+g)} e^{t_2 \mathrm{ad}(f-g)}(f+g)\,. \end{aligned}$$

Remark that the extremality condition (4) implies that $\langle p_0, h_i(\bar{q})\rangle = c$, for $i = 0, \ldots, 3$. For the seek of concision we write

$$\sigma_{ij} = \langle p_0, [h_i, h_j](\bar{q})\rangle\,. \tag{14}$$

Take U such that all the relations in inequality form which characterize the Lie brackets configuration at q_0 hold throughout $\overline{U}$. We study all trajectories of the bang-bang type described above which are contained in U (with the agreement that, when needed, we will replace U by a smaller neighborhood of q_0, which we will still denote by U). In particular we will look at Q as at a family of quadratic forms parameterized also by the point $\bar{q}$.

Choose a triple $\xi_1,\ \xi_2,\ \xi_3$ of 1-forms on M such that their restriction to U is a dual (moving) basis to $g(q),\ [f,g](q),\ -X_-(q)$ (that is: $\langle \xi_i(q), g(q)\rangle = \delta_{1i}$, $q \in U$, $i = 1, 2, 3$, and so on). Equality (9) implies that p_0 is a linear combination of $\xi_2(q)$ and $\xi_3(q)$. Moreover, the case in which p_0 is proportional to $\xi_2(q)$ can be excluded choosing T small enough, as we can easily deduce from (10). Since $p(\cdot)$ is defined up to multiplication by a positive scalar we may normalize: $p_0 = \varepsilon\xi_2(q) \pm \xi_3(q)$. Remark that

$$\varepsilon = \langle p_0, [f,g](\bar{q})\rangle = \dot{\varphi}(0) \le 0\,, \tag{15}$$

since φ is nonpositive in a right neighborhood of 0.

Set $\lambda_\star = \langle \xi_3, X_\star\rangle$, $\star = +, -, \ldots$ Let us stress that these are functions defined on U and that, by definition $\lambda_- \equiv -1$.

In the following we will evaluate the asymptotics of various quantities as the time-length of the trajectory goes to 0. To do this we will consider functions of t and q defined on $(0,T)\times U$ and we will say that $\chi(t,q)$ is of order r with respect to t (we will write $\chi(t,q)=O(t^r)$) if it is actually of order r uniformly in U as t goes to 0.

From (10) we have

$$0=t_1\langle p_0,[f,g](\bar q)\rangle+\langle p_0,O(t_1^2)\rangle=\varepsilon(t_1+O(t_1^2))+O(t_1^2)\,,$$

and thus for T small enough we can think at ε as a function of t_1 and $\bar q$. This shows that p_0 is uniquely determined up to sign by (9) and (10). Remark moreover that, as a function on $(0,T)\times U$, ε is of order 1 with respect to t_1.

From Lemma 1, since X_- is linearly independent from g and $[f,g]$ in U, we deduce that (maybe taking a smaller T and a smaller U) $\langle p(t),X_-(q(t))\rangle$ has constant sign, which necessary is -1 since this function is the second derivative of φ in the interval (t_1,t_1+t_2). Since $\langle\varepsilon\xi_2(\bar q)\pm\xi_3(\bar q),X_-(\bar q)\rangle$ depends continuously on ε, it is clear that, eventually for a further smaller T, p_0 is necessarily equal to $\varepsilon\xi_2(\bar q)+\xi_3(\bar q)$. This proves the stated uniqueness of p_0.

Another important remark is that the subspace of $\mathbf{R}^4$ defined by (12) and (13) is one-dimensional. Equivalently, let us show that

$$V=\left\{\sum_{i=0}^{3}\alpha_i h_i(\bar q)\;\middle|\;\sum_{i=0}^{3}\alpha_i=0\right\},$$

is two-dimensional. Let $\lambda\in T^*_{\bar q}M$ such that $\langle\lambda,V\rangle=0$. Then $\langle\lambda,h_i(\bar q)\rangle$ is constant with respect to $i=0,\ldots,4$, and so λ verifies the relations (9), (10) and (11). Thus λ is proportional to p_0, which means that the orthogonal space to V is one-dimensional and the remark is proved. Moreover the space of admissible α is given by the solutions of the system

$$\left\{\begin{array}{lllll}\alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & =0\,,\\ -\alpha_0 & +\alpha_1 & +\langle\xi_1(\bar q),(h_2-f)(\bar q)\rangle\alpha_2 & +\langle\xi_1(\bar q),(h_3-f)(\bar q)\rangle\alpha_3 & =0\,,\\ & & +\langle\xi_2(\bar q),(h_2-f)(\bar q)\rangle\alpha_2 & +\langle\xi_3(\bar q),(h_3-f)(\bar q)\rangle\alpha_3 & =0\,.\end{array}\right.\tag{16}$$

4.1 Case 0

From (10) and recalling that $\varepsilon=O(t_1)$, we have

$$0=t_1\langle p_0,[f,g](\bar q)\rangle+\frac{t_1^2}{2}\langle p_0,X_+(\bar q)\rangle+O(t_1^3)=\varepsilon t_1+\frac{t_1^2}{2}\lambda_++O(t_1^3).$$

Thus

$$\varepsilon=-t_1\frac{\lambda_+}{2}+O(t_1^2).\tag{17}$$

Notice that we can exclude the eventuality that $\varepsilon=0$ by choosing T small enough. Similarly from (11) we get

$$t_2=-\frac{2}{\lambda_-}(\varepsilon+t_1\lambda_+)+O(t_2^2)=-\frac{\lambda_+}{\lambda_-}t_1+O(t_1^2).$$

In particular t_2 is of the same order of t_1 and we have:

$$\begin{aligned} h_2 &= f - g - 2t_1[f,g] + O(t_1^2)\,, \\ h_3 &= f + g + 2t_2[f,g] + O(t_1^2)\,, \end{aligned}$$

and so

$$\begin{aligned} \sigma_{01} &= 2\varepsilon\,, \\ \sigma_{02} &= -2t_1\lambda_- + O(t_1^2)\,, \\ \sigma_{12} &= -2\varepsilon - 2t_1\lambda_+ + O(t_1^2)\,, \\ \sigma_{03} &= 2\varepsilon + 2t_2\lambda_- + O(t_1^2)\,, \\ \sigma_{13} &= 2t_2\lambda_+ + O(t_1^2)\,, \\ \sigma_{23} &= 2\varepsilon + 2t_2\lambda_- + 2t_1\lambda_+ + O(t_1^2)\,. \end{aligned}$$

The system (16) has the form:

$$\left\{\begin{array}{llllll} \alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & = & 0\,, \\ -\alpha_0 & +\alpha_1 & -\alpha_2 & +\alpha_3 & = & O(t_1^2)\,, \\ & & -(2t_1 + O(t_1^2))\alpha_2 & +(2t_2 + O(t_1^2))\alpha_3 & = & 0\,, \end{array}\right.$$

from which we obtain

$$\left\{\begin{array}{llll} \alpha_0 & = & -\frac{t_2}{t_1}\alpha_3 & +O(t_1)\,, \\ \alpha_1 & = & -\alpha_3 & +O(t_1^2)\,, \\ \alpha_2 & = & \frac{t_2}{t_1}\alpha_3 & +O(t_1)\,. \end{array}\right.$$

Finally, after some calculation,

$$Q = Q(\alpha_3) = 4\varepsilon\frac{t_2}{t_1}\alpha_3^2 + O(t_1^2).$$

Recalling that $\varepsilon < 0$ we have that Q is negative defined (for small T), and so $q(\cdot)$ is essential. (Indeed $\varepsilon t_2/t_1^2$ is 'exactly' a $O(1)$ in the sense that its limit as t_1 tends to 0 is a non-vanishing function on $\bar{U}$.)

This completes the proof that in the hypothesis of Case 0 locally at q_0 a bang-bang quasi optimal trajectory has no more than two switches. We remark that the choice we made between the order of + and − arcs is irrelevant since the hypotheses we where starting from are symmetric in two the signs.

4.2 Case 1

In the case considered here we can always assume, by taking U small enough, that $|\lambda_+|$ is uniformly bounded in U by any prescribed positive constant. Given a function χ of t and q defined on $(0,T)\times U$ we will find useful to write that $\chi(t,q) = \Omega(t^r)$ if $\chi(t,q) = \sum_{0\leq n\leq r}\lambda_+(q)^n O(t^{r-n})$. We remark that a simple rule of multiplication is $\Omega(t^r)\Omega(t^s) = \Omega(t^{r+s})$ and in particular $O(t^r)\Omega(t^s) = \Omega(t^{r+s})$.

Set $\pi_\star = \langle p_0, X_\star \rangle$, $\star = +, -, \ldots$ From (10) we have

$$\begin{aligned} 0 &= t_1\varepsilon + \frac{t_1^2}{2}\pi_+ + \frac{t_1^3}{6}\pi_{++} + O(t_1^4) \\ &= t_1\left(\varepsilon(1+O(t_1)) + t_1\frac{\lambda_+}{2} + t_1^2\frac{\lambda_{++}}{6} + O(t_1^3)\right)\,. \end{aligned}$$

Thus

$$\varepsilon = -t_1\frac{\lambda_+}{2} - t_1^2\frac{\lambda_{++}}{6} + \lambda_+ O(t_1^2) + O(t_1^3) = -t_1\frac{\lambda_+}{2} - t_1^2\frac{\lambda_{++}}{6} + t_1^2\Omega(t_1)\,.$$

From (11) we get similarly

$$t_2 = -t_1\frac{\lambda_+}{\lambda_-} - \frac{2\lambda_{++}}{3\lambda_-}t_1^2 + \lambda_+ O(t_1^2) + O(t_1^3) = -t_1\frac{\lambda_+}{\lambda_-} - \frac{2\lambda_{++}}{3\lambda_-}t_1^2 + t_1^2\Omega(t_1)\,.$$

Remark that, in our notations, $t_2, \varepsilon = t_1\Omega(t_1)$. We have:

$$\begin{aligned} h_2 &= f - g - 2t_1[f,g] - t_1^2 X_+ + O(t_1^3)\,, \\ h_3 &= f + g + 2t_2[f,g] + 2t_1t_2X_+ + t_1^2\Omega(t_1^2)\,, \end{aligned}$$

and so

$$\begin{aligned} \sigma_{01} &= 2\varepsilon \\ \sigma_{02} &= -2t_1\pi_- + O(t_1^2) \\ \sigma_{12} &= -2\varepsilon - 2t_1\pi_+ - t_1^2\pi_{++} + O(t_1^3) \\ \sigma_{03} &= 2\varepsilon + 2t_2\pi_- + t_1^2\Omega(t_1) \\ \sigma_{13} &= 2t_2\pi_+ + 2t_1t_2\pi_{++} + t_1^2\Omega(t_1^2) \\ \sigma_{23} &= 2\varepsilon + 2t_2\pi_- + 2t_1\pi_+ + t_1^2\pi_{++} + t_1^2\Omega(t_1)\,. \end{aligned}$$

The space of admissible α is determined by the system (16), which has the form:

$$\left\{\begin{array}{rrrrcl} \alpha_0 & +\alpha_1 & +\alpha_2 & +\alpha_3 & = & 0\,, \\ -\alpha_0 & +\alpha_1 & -(1+O(t_1^2))\alpha_2 & +(1+O(t_1^2))\alpha_3 & = & 0\,, \\ & & -(2t_1+O(t_1^2))\alpha_2 & +(2t_2+t_1^2\Omega(t_1))\alpha_3 & = & 0\,, \end{array}\right.$$

from which we obtain

$$\left\{\begin{array}{rcll} \alpha_0 & = & -\frac{t_2}{t_1}\alpha_3 & +t_1\Omega(t_1)\,, \\ \alpha_1 & = & -\alpha_3 & +O(t_1^2)\,, \\ \alpha_2 & = & \frac{t_2}{t_1}\alpha_3 & +t_1\Omega(t_1)\,. \end{array}\right. \tag{18}$$

Thus:

$$\begin{aligned} Q(\alpha_3) &= \left[2\varepsilon\frac{t_2}{t_1} + 2t_1\pi_-\frac{t_2^2}{t_1^2} + (2\varepsilon + 2t_1\pi_+ + t_1^2\pi_{++})\frac{t_2}{t_1} - (2\varepsilon + 2t_2\pi_-)\frac{t_2}{t_1}\right. \\ &\quad \left. -(2t_2\pi_+ + 2t_1t_2\pi_{++}) - (2\varepsilon + 2t_2\pi_- + 2t_1\pi_+ + t_1^2\pi_{++})\frac{t_2}{t_1}\right]\alpha_3^2 \\ &\quad +t_1^2\Omega(t_1^2) \\ &= 2\frac{t_2}{t_1}(2\varepsilon + t_1\lambda_+ + t_2\lambda_-)\alpha_3^2 + t_1^2\Omega(t_1^2) \\ &= -2t_2(\lambda_+ + t_1\lambda_{++})\alpha_3^2 + t_1^2\Omega(t_1^2) \end{aligned}$$

If Q is nonnegative defined then, since $\varepsilon \leq 0$, the following system of inequalities is satisfied:

$$\left\{ \begin{array}{rcrcl} t_2(\lambda_+ + t_1\lambda_{++}) & + & t_1^2\Omega(t_1^2) & \leq & 0\,, \\ -t_1(\lambda_+ + t_1\frac{\lambda_{++}}{3}) & + & t_1^2\Omega(t_1) & \leq & 0\,, \end{array} \right.$$

and so

$$\left\{ \begin{array}{rcrcl} t_2(\lambda_+ + t_1\lambda_{++}) & + & t_1^2\Omega(t_1^2) & \leq & 0\,, \\ -t_2(\lambda_+ + t_1\frac{\lambda_{++}}{3}) & + & t_1^2\Omega(t_1^2) & \leq & 0\,, \end{array} \right.$$

from which we deduce

$$\frac{2}{3}t_1t_2\lambda_{++} + t_1^2\Omega(t_1^2) \leq 0\,.$$

By definition of t_2 we get:

$$\lambda_{++}\underbrace{(-\frac{\lambda_+}{\lambda_-} - \frac{2\lambda_{++}}{3\lambda_-}t_1)}_{>0} + \Omega(t_1^2) \leq 0\,. \tag{19}$$

Since $|\lambda_{++}|$ is uniformly bounded away from 0 on U, we can deduce from (19) that for U and t small enough

$$\lambda_{++}(-\frac{\lambda_+}{\lambda_-} - \frac{2\lambda_{++}}{3\lambda_-}t_1) < 0\,,$$

and so a necessary condition for Q to be nonnegative defined (and even more so a necessary condition for $q(\cdot)$ to be quasi optimal) is that $\lambda_{++} < 0$. In particular (recall that $\lambda_- < 0$) if

$$(\operatorname{sign} g \wedge [f,g] \wedge X_-(q_0))(\operatorname{sign} g \wedge [f,g] \wedge X_{++}(q_0)) < 0\,,$$

(that is: if the orientation of the two triples of vectors does not coincide) then a $-+-+$ trajectory of small enough length contained in U is not quasi optimal. If the above signs coincide then they are different for the same system with reversed time, that is, where f and g are substituted, respectively, by $-f$ and $-g$. Since the time reversed of a quasi optimal trajectory of the old system is quasi optimal for the new one we have that a $+-+-$ trajectory of the initial system of small enough length contained in U is not quasi optimal.

We conclude that a short enough quasi optimal bang-bang trajectory lying in U has no more than three switches, in both Cases 0 and 1. In particular we have proved that a short quasi optimal trajectory in U has no chattering control.

4.3 Sharpness of the result

The bound given in Case 0 is clearly sharp, as we can realize by a purely dimensional reasoning. To investigate whether the extra switch that we add in Case 1 is needed or not we refer to the sufficiency condition for optimality proved in [4, Theorem 2.6]: the same reasoning applied to the present case implies that if the quadratic form Q is positive defined, then the corresponding trajectory

$q : [0,T] \to M$ is locally time-optimal, in the $\mathcal{C}^0$ sense, that is: there exists a neighborhood U of the graph $\{(t, q(t)) \mid t \in [0,T]\}$ in $[0,T] \times M$ such that q is time-optimal between all the admissible trajectories whose graph is contained in U. Our aim is to use this result to show that the bound we gave is sharp in the following sense:

Lemma 2 *Let $\mathbb{J}^4_{q_0}$ be the space of 4-jets at q_0 of pairs of smooth vector fields on M and let $C_1 \subset \mathbb{J}^4_{q_0}$ be the set of all $J^4_{q_0}(f,g)$ such that (f,g) is of type 1 at q_0. Then it exists an open nonempty subset A_1 of C_1 such that if the 4-jet of (f,g) belongs to A_1 then for any $T > 0$ it exists a trajectory with three switches, passing through q_0 and of time length smaller than T, which is locally time-optimal in the $\mathcal{C}^0$ sense.*

It is convenient to fix the point q_0 itself as the first switch and to invert the order of $+$ and $-$ arcs. We will still use the notation $O(t^\gamma)$, $\gamma > 0$, to denote a function of $t \in (0,T)$ which is of order t^γ as t goes to 0. The equations for the switching times are:

$$
\begin{aligned}
0 &= \langle p_0, g(q_0) \rangle, && (20)\\
0 &= \langle p_0, e^{t_1 \operatorname{ad}(f-g)} g(q_0) \rangle, && (21)\\
0 &= \langle p_0, e^{t_1 \operatorname{ad}(f-g)} e^{t_2 \operatorname{ad}(f+g)} g(q_0) \rangle. && (22)
\end{aligned}
$$

In terms of $\xi_1,\ \xi_2,\ \xi_3$, defined as before, we have

$$p_0 = \varepsilon \xi_2(q_0) + \xi_3(q_0)\,,$$

with $\varepsilon \geq 0$. Remark that now

$$
\begin{aligned}
h_0 &= f + g\,,\\
h_1 &= f - g\,,\\
h_2 &= e^{t_1 \operatorname{ad}(f-g)}(f+g)\,,\\
h_3 &= e^{t_1 \operatorname{ad}(f-g)} e^{t_2 \operatorname{ad}(f+g)}(f-g)\,.
\end{aligned}
$$

We follow the same procedure as in the previous subsections: we use (21) and (22) to establish the asymptotics of ε and t_2; from (12) and (13) we find the relations among the α_i and we compute Q. The calculations are here more lengthy since the first two computable orders of Q —order $t_1^{3/2}$ and order t_1^2— both annihilate.

In details: from (21) and (22) we have

$$\varepsilon = -t_1 \frac{\pi_-}{2} - t_1^2 \frac{\pi_{--}}{6} + O(t_1^3), \tag{23}$$

and

$$t_2^2 = -3 t_1 \frac{\lambda_-}{\lambda_{++}} + O(t_1^{3/2})\,.$$

We can now compute

$$
\begin{aligned}
h_2 &= f+g+2t_1[f,g]+t_1^2X_- + O(t_1^3)\,,\\
h_3 &= f-g-2t_2[f,g]-t_2^2X_+ - 2t_1t_2X_- - \frac{t_2^3}{3}X_{++} + O(t_1^2)\,,
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{01} &= -2\varepsilon\,,\\
\sigma_{02} &= 2t_1\pi_= + t_1^2\pi_{+-} + O(t_1^3)\,,\\
\sigma_{12} &= 2\varepsilon + 2t_1\pi_- + t_1^2\pi_{--} + O(t_1^3)\,,\\
\sigma_{03} &= -2\varepsilon - 2t_2\pi_+ - t_2^2\pi_{++} - 2t_1t_2\pi_{+-} - \frac{t_2^3}{3}\pi_{+++} + O(t_1^2)\,,\\
\sigma_{13} &= -2t_2\pi_- - t_2^2\pi_{-+} - 2t_1t_2\pi_{--} - \frac{t_2^3}{3}\pi_{-++} + O(t_1^2)\,,\\
\sigma_{23} &= \sigma_{03} - \sigma_{12} + 2\varepsilon - 2t_1t_2^2\pi_\times + O(t_1^{5/2})\,,
\end{aligned}
$$

where

$$\pi_\times = \langle p_0, [[f,g],X_+](q_0)\rangle\,.$$

The space on which Q is defined turns out to be described by:

$$
\begin{aligned}
\alpha_0 &= -\alpha_2 + O(t_1^{3/2})\,,\\
\alpha_1 &= \left(-\frac{t_1}{t_2} + t_1\frac{\delta_+}{2} + \frac{t_1^2}{t_2}\frac{\delta_-}{2} + \frac{t_1t_2}{2}\left(\frac{\delta_{++}}{3} - \frac{\delta_+^2}{2} - \gamma_+\right)\right)\alpha_2 + O(t_1^2)\,,\\
\alpha_3 &= \left(\frac{t_1}{t_2} - t_1\frac{\delta_+}{2} - \frac{t_1^2}{t_2}\frac{\delta_-}{2} - \frac{t_1t_2}{2}\left(\frac{\delta_{++}}{3} - \frac{\delta_+^2}{2}\right)\right)\alpha_2 + O(t_1^2)\,.
\end{aligned}
$$

where

$$
\begin{aligned}
\gamma_+ &= \langle \xi_1(q_0), X_+(q_0)\rangle\,,\\
\delta_\star &= \langle \xi_2(q_0), X_\star(q_0)\rangle\,,\quad \star = +,-,\dots
\end{aligned}
$$

Finally we get

$$
\begin{aligned}
Q = Q(\alpha_2) = t_1^2t_2\Bigg(&\lambda_-\delta_+^2 - \lambda_{-+}\delta_+ + \lambda_-\gamma_+ + \frac{\lambda_{-++} + \lambda_{++}\delta_- - \lambda_-\delta_{++}}{3}\\
&-2\lambda_\times - \frac{2}{9}\frac{\lambda_{++}\lambda_{--}}{\lambda_-}\Bigg)\alpha_2^2 + O(t_1^3)\,.
\end{aligned}
$$

To conclude the proof of Lemma 2 it suffices to exhibit a system on $\mathbf{R}^3$ for which at the point $q_0 = 0$ the Lie brackets configuration is of type 1 and the following sign conditions hold: $\lambda_- < 0$, $\lambda_{++} > 0$ and

$$\lambda_-\delta_+^2 - \lambda_{-+}\delta_+ + \lambda_-\gamma_+ + \frac{\lambda_{-++} + \lambda_{++}\delta_- - \lambda_-\delta_{++}}{3} - 2\lambda_\times - \frac{2}{9}\frac{\lambda_{++}\lambda_{--}}{\lambda_-} > 0\,.$$

This is the case, for instance, of the control system:

$$f(x,y,z) = \begin{pmatrix} 0 \\ -x \\ -\frac{x^2}{4} - \frac{x^3}{4} + \frac{y}{2} + y^2 - z \end{pmatrix}, \quad g(x,y,z) = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

for which we get $\lambda_- = -1$, $\lambda_{++} = 2$ and

$$Q = Q(\alpha_2) = \frac{17}{6} t_1^2 t_2 \alpha_2^2 + O(t_1^3)\,.$$

5 Allowing singular arcs

Let now $q : (0,T) \to M$ be a quasi optimal trajectory with no preliminary restriction on the structure of its corresponding control function $u(\cdot)$. It is known from Proposition 1 in [1] that $u(\cdot)$, after eventually a modification on a set of measure zero, is $\mathcal{C}^\infty$ on a open dense subset O of $(0,T)$. Assume that O is maximal (also with respect to further modifications of u on sets of measure zero). An *arc* is a piece of trajectory corresponding to a connected component of O; it is *bang* if u takes value in $\{-1,1\}$ on it, whereas it is *singular*. We will use the word arc also to refer to the connected component of O itself. Remark that on an arc also φ is $\mathcal{C}^\infty$ and all its derivatives can be computed iterating (8). We say that two distinct arcs (s_1,s_2) and (t_1,t_2) are *concatenated* if $s_2 = t_1$ or $s_1 = t_2$. Let $\Sigma = \partial O$. We say that two distinct points of Σ are *subsequent* if the open interval which they identify does not intersect Σ (that is, by density of O, if it is an arc).

We start from the situation in which there is no bang arc. If this is the case, then the switching function is identically equal to zero: otherwise there would be singular arc S with a proper nonempty intersection with $\{\tau|\varphi(\tau) \neq 0\}$. Let t be in the boundary of $\{\tau|\varphi(\tau) \neq 0\}$ and in the interior of S. Since $\varphi^{(n)}(t) = 0$, $n \geq 0$, it follows from the nondegeneracy conditions of cases 0 or 1 that $p(t)$ is equal to 0, which is impossible. Thus φ is identically equal to zero. We want to deduce that the trajectory is made of a single singular arc, by proving that if $\varphi \equiv 0$ on an open interval I, then u is smooth on I. Indeed: both $\dot\varphi$ and its further derivatives are also identically equal to zero on I and so $p(t)$ is orthogonal to both $g(q(t))$ and $[f,g](q(t))$ and, for almost all $t \in I$,

$$\langle p(t), [f,[f,g]](q(t))\rangle + u(t)\langle p(t), [g,[f,g]](q(t))\rangle = 0\,. \tag{24}$$

We remark that in both Cases 0 and 1

$$\mathrm{span}\{g(q), [f,g](q), [f,[f,g]](q), [g,[f,g]](q)\} = T_qM$$

for every $q \in U$. Thus $\langle p(t), [g,[f,g]](q(t))\rangle \neq 0$ for every t for which (24) holds, otherwise $p(t) \neq 0$ would annihilate $T_{q(t)}M$. If $\langle p(\bar t), [g,[f,g]](q(\bar t))\rangle = 0$ for some $\bar t \in I$, however, we would have that near $\bar t$ the function $t \mapsto \langle p(t), [f,[f,g]](q(t))\rangle$

is bounded away from 0 and consequently $|u(t)| > 1$ for some t at which (24) holds. Thus, for every $t \in I$, $\langle p(\bar{t}), [g,[f,g]](q(\bar{t}))\rangle \neq 0$ and

$$u(t) = -\frac{\langle p(t), [f,[f,g]](q(t))\rangle}{\langle p(t), [g,[f,g]](q(t))\rangle}\,.$$

Substituting the last expression in the Hamiltonian, we find that $p|_I$ is a solution of the smooth (autonomous) Hamiltonian system generated by the Hamiltonian

$$h(p) = \langle p, f\rangle - \frac{\langle p, [f,[f,g]]\rangle}{\langle p, [g,[f,g]]\rangle}\langle p, g\rangle\,.$$

If no bang arc $(\tau_1, \tau_2) \subset\subset (0,T)$ exists then, by the same reasoning, the trajectory is a concatenation of at most a bang a singular and a bang arc. Let now $(\tau_1, \tau_2) \subset\subset (0,T)$ be a bang arc. We can associate to it the smaller point t_2 of Σ such that: $t_2 \geq \tau_2$, t_2 is the upper bound of a bang arc and is not the lower bound of any bang arc (this is possible since the results of the previous section exclude, for T small enough, the existence of an infinite sequence of subsequent bang arcs, as already remarked). Analogously we can define $t_1 \leq \tau_1$. Assume that $t_2 \neq T$. Then $p(t_2)$ is orthogonal to both $g(q(t_2))$ and $[f,g](q(t_2))$. Indeed by definition of t_2 in each of its (right) neighborhoods lies the interior point of a singular arc or an entire bang arc, on which $\varphi(t)$ necessarily assumes a zero and a maximum or minimum. By continuity we deduce the stated orthogonality. As follows from Lemma 1, we can fix U and T such that the signs of $\langle p(\cdot), X_-(q(\cdot))\rangle$ and $\langle p(\cdot), X_{++}(q(\cdot))\rangle$ do not vary along the trajectory. In particular we can exclude that the bang arc having t_2 as upper bound corresponds to control -1, since $\langle p(\cdot), X_-(q(\cdot))\rangle$ is the second derivative of $\varphi(\cdot)$ along the arc and the first derivative takes value zero in t_2- (and in an internal point of the arc, corresponding to a maximum of φ, since the arc itself is compactly contained in $(0,T)$). Thus the Lie brackets configuration is of type 1 and t_2 is the endpoint of a $+$ arc. Moreover it can be easily proved that the sign of $\langle p(\cdot), X_{++}(q(\cdot))\rangle$ along the trajectory is -1. Analogous considerations could be carried out in t_1 in the case $t_1 \neq 0$, leading up to the opposite sign of $\langle p(\cdot), X_{++}(q(\cdot))\rangle$. Thus $t_1 = 0$. By the same reasonings there cannot be any bang arc compactly contained in the interval (t_2, T). Thus (t_2, T) is a singular arc itself or the union of a singular and a bang arc. Up to now we proved that a short enough quasi optimal trajectory lying in U is the concatenation of at most four bang arcs, a singular arc and a bang arc. Actually if the considered trajectory contains a singular arc, an extra quasi optimality condition is fulfilled: generalized Legendre condition (see [7] for a formulation and [2] for a complete mathematical proof). In the present situation this condition states that $\langle p(\cdot), X_-(q(\cdot))\rangle$ is negative along the singular arc and so along the entire trajectory. Thus we exclude that a trajectory has both a singular and a compactly contained bang arc corresponding to control -1. Finally the two 'maximal' non bang-bang concatenations are $-{+}\mathrm{S}\pm$ or $\pm\mathrm{S}{+}-$ (with the choice between $+$ and $-$ made in the formulation of Case 1).

We include here a result which is in the same spirit of the reasonings above and which improves Proposition 2 in [1].

Lemma 3 *Take a finite dimensional manifold M and two smooth vector fields f and g on M. Let $X_+,\ X_-,\dots$ be defined as usual. We will also write $X_{(m+)}$ for $X_{\underbrace{+\dots+}_{m\ \text{times}}}$ (and $X_{(m-)}$ for the natural counterpart). Assume that the system*

$$\dot q = f(q) + ug(q), \quad |u| \le 1\,, \tag{25}$$

is such that: for each $q_0 \in M$ and for $\star = +, -$, we have:

$$T_{q_0}M = \mathrm{span}\{g(q_0), [f,g](q_0), X_\star(q_0), \dots, X_{(m\star)}(q_0), \dots\} \stackrel{\mathrm{def}}{=} V_\star(q_0)\,.$$

Let $q : (0,T) \to M$ be an extremal trajectory of the system above such that the corresponding control u verifies $|u| = 1$ a.e. and let Σ be the set of discontinuities of u (not avoidable by changing u on a set of measure zero). Then either Σ is discrete or it contains a sequence of infinitely many subsequent isolated points.

Proof. Let O be the maximal open dense subset of $(0,T)$ on which u, after modification on a set of measure zero, is smooth. Clearly u is constant on any arc, i.e. any connected component of O.

Assume by contradiction that neither Σ is discrete, nor it contains an infinite sequence of subsequent isolated points. To each (bang) arc (τ_1, τ_2) we can associate the smaller point of Σ which is larger or equal than τ_2 and which is not isolated, unless it exists a finite sequence of subsequent isolated points including τ_2 and T. Denote by A the set of all points which can be associated as described to some arc. By our assumptions A is nonempty and preperfect, i.e. each point of A is a density point for A. We give a partition $A = A_+ \cup A_-$ by defining $A_\star$, $\star = +, -$, as the set of points of A for which the control u on a left neighborhood of it has constant sign $\star 1$. We have that it exists $\star \in \{+, -\}$ and it exists a subset B of $A_\star$ which is nonempty and preperfect (indeed: if $a \in A_+$ is not a density point for A_+ then it exists a neighborhood U of a such that $(U \setminus \{a\}) \cap A$ is a preperfect nonempty subset of A_- and thus either A_+ is nonempty and preperfect either A_- has a preperfect nonempty subset).

Let $p(\cdot)$ be a covector trajectory associated to $q(\cdot)$. To complete the argument we want to prove that at each point $\tau \in B$ the covector $p(\tau)$ annihilates $V_\star(q(\tau))$. It is clear that $\varphi(\tau) = \langle p(\tau), g(q(\tau))\rangle = 0$ for each $\tau \in \Sigma$. Since on the interval between two subsequent points of Σ there is an extreme point of $\varphi(t)$ and since $\varphi'(t) = \langle p(t), [f,g](q(t))\rangle$, then we have that, by continuity, $\langle p(\tau), [f,g](q(\tau))\rangle = 0$ for each $\tau \in A$. Reasoning similarly and using the preperfectness of B we prove by induction on m that $\langle p(\tau), X_{(m\star)}(q(\tau))\rangle = 0$ for each $\tau \in B$. $\square$

References

[1] A. A. Agrachev, On Regularity Properties of Extremals Controls, *Journal of Dynamical and Control Systems*, **1**, No. 3 (1995), pp. 319-324.

[2] A. A. Agrachev and R. V. Gamkrelidze, A Second Order Optimality Principle for a Time-optimal Problem, *Math. USSR Sbornik*, **29**, No. 4 (1976), pp. 547-576.

[3] A. A. Agrachev and R. V. Gamkrelidze, Symplectic Geometry for Optimal Control, in *Nonlinear Controllability and Optimal Control* (Hector J. Sussmann, ed.), Pure and Applied Mathematics, Vol. 133, Marcel Dekker, 1990.

[4] A. A. Agrachev, G. Stefani and P. Zezza, Strong Optimality for a Bang-bang Trajectory, *SIAM J. Control and Optimization*, to appear.

[5] A. Bressan, The Generic Local Time-optimal Local Stabilizing Controls in Dimension 3, *SIAM J. Control and Optimization*, **24**, No. 2 (1986), pp. 177-190.

[6] A. T. Fuller, Study of an Optimum non Linear System, *J. Electronics Control*, **15** (1963), pp. 63-71.

[7] H. J. Kelley, R. E. Kopp and H. Gardner Moyer, Singular Extremals, Topics in Optimization, Academic Press, New York, 1967, pp. 63-101. MR 35 No. 5996.

[8] I. A. K. Kupka, The Ubiquity of Fuller's Phenomenon, in *Nonlinear Controllability and Optimal Control* (Hector J. Sussmann, ed.), Pure and Applied Mathematics, Vol. 133, Marcel Dekker, 1990.

[9] H. Schättler, On the Local Structure of Time-optimal Bang-bang Trajectories in $\mathbf{R}^3$, *SIAM J. Control and Optimization*, **26**, No. 1 (1988), pp. 186-204.

[10] H. Schättler, The Local Structure of Time-optimal Trajectories in Dimension Three Under Generic Conditions, *SIAM J. Control and Optimization*, **26**, No. 4 (1988), pp. 899-918.

[11] H. Schättler, Regularity Properties of Optimal Trajectories: Recently Developed Techniques, in *Nonlinear Controllability and Optimal Control* (Hector J. Sussmann, ed.), Pure and Applied Mathematics, Vol. 133, Marcel Dekker, 1990.

[12] H. J. Sussmann, Time-Optimal Control in the Plane, in *Feedback Control of Linear and Nonlinear Systems*, Lecture Notes in Control and Information Sciences, **39**, Springer-Verlag, Berlin 1985, pp. 244-260.

[13] H. J. Sussmann, Envelopes, Conjugate Points and Optimal Bang-bang Extremals, in *Proc. 1985 Paris Conf. on Nonlinear Systems* (M. Fliess and M. Hazewinkel, eds.), D. Reidel Publishing Co., Dordrechet, 1986.

[14] H. J. Sussmann, A weak regularity for real anlalytic optimal control problems, *Revista Matemática Iberoamericana*, **2**, No. 3 (1986), pp. 307-317.

[15] H. J. Sussmann, Envelopes, High Order Optimality Conditions and Lie Brackets, in *Proceedings of the 28th IEEE Conference on Decision and Control, Tampa, Florida*, pp.1107-1112, 1989.

[16] M. I. Zelikin, V. F. Borisov, Theory of chattering control. With applications to astronautics, robotics, economics, and engineering, Systems and Control: Foundations and Applications, Birkhuser Boston, 1994.