Geometry of Jacobi Curves I

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Abstract

Jacobi curves are far going generalizations of the spaces of "Jacobi fields" along Riemannian geodesics. Actually, Jacobi curves are curves in the Lagrange Grassmannians. In our paper we develop differential geometry of these curves which provides basic feedback or gauge invariants for a wide class of smooth control systems and geometric structures. Two principal invariants are: the *generalized Ricci curvature*, which is an invariant of the parametrized curve in the Lagrange Grassmannian providing the curve with a natural projective structure, and a *fundamental form*, which is a degree 4 differential on the curve. The so-called rank 1 curves are studied in greater detail. Jacobi curves of this class are associated to systems with scalar controls and to rank 2 vector distributions.

In the forthcoming second part of the paper we will present the comparison theorems (i.e., the estimates for the conjugate points in terms of our invariants) for rank 1 curves and introduce an important class of "flat curves".

Key words: Lagrange Grassmannian, Jacobi curve, symplectic invariants, feedback invariants, cross-ratio.

1 Introduction

Suppose M is a smooth n-dimensional manifold and $\pi : T^*M \to M$ is the cotangent bundle to M. Let H be a codimension 1 submanifold in T^*M such that H is transversal to T_q^*M, $\forall q \in M$; then $H_q = H \cap T_q^*M$ is a smooth hypersurface in T_q^*M. Let ς be the canonical Liouville form on T_q^*M, $\varsigma_\lambda = \lambda \circ \pi_*$, $\lambda \in T^*M$, and $\sigma = -d\varsigma$ be the standard symplectic structure on T^*M; then $\sigma|_H$ is a corank 1 closed 2-form. The kernels of $(\sigma|_H)_\lambda$, $\lambda \in H$ are transversal to T_q^*M, $q \in M$; these kernels form a line distribution in H and define a *characteristic 1-foliation* $\mathcal{C}$ of H. Leaves of this foliation are *characteristic curves* of $\sigma|_H$.

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Suppose γ is a segment of a characteristic curve and O_γ is a neighborhood of γ in H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is in fact a symplectic manifold endowed with a symplectic structure $\bar{\sigma}$ induced by $\sigma|_H$. Let $\phi : O_\gamma \to N$ be the canonical factorization; then $\phi(H_q \cap O_\gamma)$, $q \in M$, are Lagrangian submanifolds in N. Let $L(T_\gamma N)$ be the Lagrangian Grassmannian of the symplectic space $T_\gamma N$, i.e. $L(T_\gamma N) = \{\Lambda \subset T_\gamma N : \Lambda^\angle = \Lambda\}$, where $D^\angle = \{e \in T_\gamma N : \bar{\sigma}(e, D) = 0\}$, $\forall D \subset T_\gamma N$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*(T_\lambda H_{\pi(\lambda)}), \quad \lambda \in \gamma,$$

from γ to $L(T_\gamma N)$.

Jacobi curves are curves in the Lagrange Grassmannians. They are invariants of the hypersurface H in the cotangent bundle. In particular, any differential invariant of the curves in the Lagrange Grassmannian by the action of the linear Symplectic Group (i.e., any symplectic invariant) produces a well-defined function on H.

To make things clear it is not worse to give a coordinate version of the construction of Jacobi curve. In the neighborhood O_γ choose coordinates $(x_0, x_1, \dots, x_{2n-2})$ such that the characteristic curves of $\sigma|_H$ are the straight lines parallel to the x_0-axis (here we do not care about the fact that H come from the linear fiber bundle T^*M, we forget about the linear structure of the fibers). In these coordinates the sets $H_{\pi(\lambda)}$ are some $(n-1)$-dimensional submanifolds of $\mathbb{R}^{2n-1}$. For any $\lambda \in \gamma$ take projection (parallel to x_0-axis) of the spaces $T_\lambda H_{\pi(\lambda)}$ to the hyperplane $\{x_0 = c\}$ for some c. Then we obtain a curve of $(n-1)$-dimensional subspaces in the $(2n-2)$-dimensional linear space. The restriction of the form σ to $\{x_0 = c\}$ provides this space with symplectic structure and the obtained curve is a curve of Lagrangian subspaces w.r.t. this structure. This curve is exactly the Jacobi curve.

Set $W = T_\gamma N$ and note that the tangent space $T_\Lambda L(W)$ to the Lagrangian Grassmannian at the point Λ can be naturally identified with the space of quadratic forms on the linear space $\Lambda \subset W$. Namely, take a curve $\Lambda(t) \in L(W)$ with $\Lambda(0) = \Lambda$. Given some vector $l \in \Lambda$, take a curve $l(\cdot)$ in W such that $l(t) \in \Lambda(t)$ for all t and $l(0) = l$. Define the quadratic form $q_{\Lambda(\cdot)}(l) = \frac{1}{2}\bar{\sigma}(\frac{d}{dt}l(0), l)$. Using the fact that the spaces $\Lambda(t)$ are Lagrangian, i.e. $\Lambda(t)^\angle = \Lambda(t)$, it is easy to see that the form $q_{\Lambda(\cdot)}(l)$ depends only on $\frac{d}{dt}\Lambda(0)$. So, we have the map from $T_\Lambda L(W)$ to the space of quadratic forms on Λ. A simple counting of dimension shows that this mapping is a bijection. Below we use the just described identification of tangent vectors to $L(W)$ with quadratic forms without a special mentioning.

Proposition 1 *Tangent vectors to the Jacobi curve J_γ at a point $J_\gamma(\lambda)$, $\lambda \in \gamma$, are equiva-*

lent (under linear substitutions of variables in the correspondent quadratic forms) to the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} \subset T^*_{\pi(\lambda)}M$ at the point λ.

Sketch of proof. In our local study we may assume without lack of generality that H is a regular level set of a smooth function h on T^*M. Then γ is a trajectory of the Hamiltonian vector field $\vec{h}$ defined by the identity $\vec{h}\rfloor\sigma = dh$. Let $t \mapsto \gamma(t)$ be a parametrization of γ defined by the Hamiltonian system $\frac{d}{dt}\gamma = \vec{h}(\gamma)$, $\gamma(0) = \lambda$. Given $l \in \phi_*(T_\lambda H_{\pi(\lambda)})$, take a vector field ℓ on H such that $\ell(\gamma(t)) \in T_{\gamma(t)}H_{\pi(\gamma(t))}$, $\phi_*\ell(\lambda) = l$. Simple calculations show that $\frac{d}{dt}\phi_*\ell(\gamma(t)) = \phi_*[\vec{h},\ell](\gamma(t))$. Hence

$$\frac{d}{dt}J_\gamma|_{t=0}(l) = \bar{\sigma}(\frac{d}{dt}\phi_*\ell(\gamma(t))|_{t=0}, l) = \sigma([\vec{h},\ell](\lambda), \ell(\lambda)).$$

Now we rewrite the last formula in coordinates. Let $q = (q^1, \dots, q^n)$ be local coordinates in M and $p = (p_1, \dots, p_n)$ be induced coordinates in the fiber of T^*M so that $\varsigma = \sum\limits_{i=1}^n p_i dq^i$, $\sigma = \sum\limits_{i=1}^n dp_i \wedge dq^i$. Then $\vec{h} = \sum\limits_{i=1}^n \left(\frac{\partial h}{\partial p_i}\frac{\partial}{\partial q^i} - \frac{\partial h}{\partial q^i}\frac{\partial}{\partial p_i}\right)$, $\ell = \sum\limits_{i=1}^n \ell^i \frac{\partial}{\partial p_i}$, $l = (\ell^1(\lambda), \dots, \ell^n(\lambda))$. We have

$$\sigma([\vec{h},\ell](\lambda), \ell(\lambda)) = l^* \frac{\partial^2 h}{\partial p^2} l.$$

Quadratic form $l \mapsto l^*\frac{\partial^2 h}{\partial p^2}l$ is exactly the "second fundamental form" of the hypersurface $H_{\pi(\lambda)} = h^{-1}(h(\lambda)) \cap T^*_{\pi(\lambda)}M$ in $T^*_{\pi(\lambda)}M$. $\square$

In particular, the velocity of J_γ at λ is a sign-definite quadratic form if and only if the hypersurface $H_{\pi(\lambda)}$ is strongly convex at λ.

A similar construction can be done for a submanifold of codimension 2 in T^*M. Namely, let H be a transversal to fibers codimension 2 submanifold in T^*M. In general, characteristic curves do not fill the whole submanifold H; they are concentrated in the characteristic variety consisting of the points, where $\sigma|_H$ is degenerate. In our local study we may always assume that H is orientable and let Ω be a volume form on M. Then $\bigwedge^{n-1}\sigma|_H = a\Omega$, where a is a smooth function on H. We set

$$C_H = \{\lambda \in H : a(\lambda) = 0,\ (d_\lambda a \bigwedge^{n-1}\sigma|_\lambda)|_H \neq 0\}.$$

Assume that $C_H \neq \emptyset$. Then C_H is a codimension 1 submanifold of H and $\sigma|_{C_H}$ is a 2-form of corank 1 on C_H. Indeed, $\forall\lambda \in C_H$, $\ker\sigma_\lambda|_H$ is a 2-dimensional subspace in $T_\lambda H$, which is transversal to $T_\lambda C_H$, and we have $\ker\sigma_\lambda|_{C_H} = \ker\sigma_\lambda|_H \cap T_\lambda C_H$.

The characteristic curves of $\sigma|_{C_H}$ form a 1-foliation $\mathcal{C}$ of C_H. Let γ be a segment of a characteristic curve and O_γ be a neighborhood of γ in H such that $N = O_\gamma/(\mathcal{C}|_{O_\gamma})$ is a well-defined smooth manifold. The quotient manifold N is a symplectic manifold endowed with a symplectic structure $\bar\sigma$ induced by $\sigma|_{C_H}$. Let $\phi : O_\gamma \to N$ be the canonical factorization. It is easy to check that $\phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big)$ is a Lagrangian subspace of the symplectic space $T_{\phi(\lambda)}N$, $\forall \lambda \in O_\gamma$. *Jacobi curve of the characteristic curve* γ is the mapping

$$\lambda \mapsto \phi_*\big((T_\lambda H_{\pi(\lambda)} + \ker \sigma_\lambda|_H) \cap T_\lambda C_H\big), \quad \lambda \in \gamma,$$

from γ to $L(T_\gamma N)$.

We are mainly interested in submanifolds that are dual objects to smooth control systems. Here we call a smooth control system any submanifold $V \subset TM$, transversal to fibers. Let $V_q = V \cap T_qM$; The "dual" *normal variety* H^1 and *abnormal variety* H^0 are defined as follows:

$$H^1 = \bigcup_{q\in M} \{\lambda \in T_q^*M : \exists v \in V_q,\ \langle \lambda, v\rangle = 1,\ \langle \lambda, T_vV_q\rangle = 0\},$$

$$H^0 = \bigcup_{q\in M} \{\lambda \in T_q^*M \setminus 0 : \exists v \in V_q,\ \langle \lambda, v\rangle = \langle \lambda, T_vV_q\rangle = 0\}.$$

These varieties are not, in general, smooth manifolds; they may have singularities, which we do not discuss here. Anyway, one can obtain a lot of information on the original system just studying smooth parts of H^1, H^0.

One of the varieties H^1, H^0 can be empty. In particular, if $V_q = \partial W_q$, where W_q is a convex set and $0 \in intW_q$, then $H^0 = \emptyset$. Moreover, in this case the Liouville form never vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^1}$, and any characteristic curve γ has a canonical parametrization by the rule $\langle \varsigma, \dot\gamma\rangle = 1$. If subsets $V_q \subset T_qM$ are conical, $\alpha V_q = V_q$, $\forall \alpha > 0$, then, in contrast to the previous case, $H^1 = \emptyset$ and ς vanishes on the tangent lines to the characteristic curves of $\sigma|_{H^0}$. The characteristic curves are actually unparametrized.

Characteristic curves of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) are associated with normal (abnormal) extremals of the control system V. In [1,2] Jacobi curves of extremals were defined in purely variational way in terms of the original control system and in a very general setting (singularities included), see also [6]. The introduced here Jacobi curves of characteristic curves of $\sigma|_{H^1}$ ($\sigma|_{H^0}$) coincide with Jacobi curves of the extremals, associated with these characteristic curves, in the following important cases:

1. If H^1 has codimension 1 in T^*M. It occurs, for example, if subsets V_q are compact $\forall q \in M$;

2. If H^0 has codimension 1 in T^*M, but $H^1 = \emptyset$. It occurs, for example, if for any q subset V_q is conical but does not contain a 2-dimensional linear space;

3. If H^1 has codimension 2. It occurs, for example, if for any q subset V_q is affine line in T_qM, not containing the origin;

4. H^0 has codimension 2. It occurs, for example , if V_q are 2-dimensional linear spaces, i.e., subsets V_q define rank 2 vector distribution on M, or if $V_q = D_q \cap \partial W_q$, where D_q is 2-dimensional linear space and W_q is a convex set such that $0 \in int W_q$.

Jacobi curves associated with extremals of given control system are not arbitrary curves of Lagrangian Grassmannian but they inherit special features of the control system. The rank of the "second fundamental form" of the submanifolds H_q^1 and H_q^0 of T_q^*M at any point is no greater than $\dim V_q$. Indeed, let $\lambda \in H_q^1$; then $\lambda \in (T_v V_q)^\perp$, $\langle \lambda, v \rangle = 1$, for some $v \in V_q$. We have $\lambda + (T_v V_q + \mathbb{R}v)^\perp \subset H_q^1$. So λ belongs to an affine subspace of dimension $n - \dim V_q - 1$, which is contained in H_q^1. For $\lambda \in H_q^0$, $\exists v \in T_q M$ such that $\lambda \in (T_v V_q)^\perp$, $\langle \lambda, v \rangle = 0$. Then linear subspace $(T_v V_q + \mathbb{R}v)^\perp$ is contained in H_q^0. It follows that the second fundamental forms of H_q^1 and H_q^0 have rank no greater than $(\dim V_q - \operatorname{codim} H^1 + 1)$ and $(\dim V_q - \operatorname{codim} H^0 + 1)$ correspondently.

In the cases 1 and 2 the velocity of the Jacobi curve $\lambda \mapsto J_\gamma(\lambda)$, $\lambda \in \gamma$, associated with the extremal γ, has rank no greater than $\dim V_{\pi(\lambda)}$ (see Prop. 1). The same is true for the Jacobi curves of the extremals in the cases 3 and 4, although Prop. 1 cannot be directly applied.

Dimension of V_q is the number of inputs or control parameters in the control system. Less inputs means more "nonholonomic constraints" on the system. It happens that the rank of velocity of any Jacobi curve generated by the system never exceeds the number of inputs.

Note that by construction these Jacobi curves are feedback invariants of the control system (i.e., they do not depend on a parametrization of the sets V_q). Hence any symplectic invariants of the Jacobi curves, associated with extremals, defines a function on appropriate submanifold of T^*M that is a feedback invariant of the control system. In this way the problem of finding feedback invariants of control systems can be reduced to the much more treatable problem of finding symplectic invariants of certain curves in the Lagrange Grassmannian.

A curve in the Lagrange Grassmannian is called regular, if its velocity at any point is a non-degenerated quadratic forms. Regular curves were studied in [1], where notions of the derivative curve and the curvature operator were introduced. Actually the derivative curves of Jacobi curves, associated with the hypersurface H, provide a canonical connection on the cotangent bundle. If H is a spherical bundle of a Riemannian manifold, then this connection is just the Levi-Civita connection. The curvature operator of the Jacobi curve is intimately related to the curvature tensor of the canonical connection.

In the present paper we develop general theory of curves in the Lagrangian Grassmannian. The first steps in this direction were made in [3]. It makes sense to restrict ourselves to studying so-called monotonic (i.e, nondecreasing or nonincreasing) curves. The curve in Lagrangian Grassmannian is called nondecreasing (nonincreasing), if the velocity at any its point is nonpositive (correspondingly, nonnegative) quadratic form. Jacobi curve associated with the extremal of finite Morse index is automatically monotonic.

This paper is organized as follows. In section 2 we give the general construction of the derivative curve and introduce two principal discrete characteristic of the curves in the Lagrange Grassmannian: the rank and the weight. In particular, regular curves have maximal rank and minimal weight. Derivative curve is defined for any curve of the finite weight. In section 3 we define the curvature operator and show its role for the regular curves.

In section 4 we study the *cross-ratio* of four points and an infinitesimal cross-ratio of two tangent vectors at two distinct points in the Lagrange Grassmannian. The last one leads to an intrinsic pairing $V_0, V_1 \mapsto \langle V_0 \mid V_1\rangle$, $V_i \in T_{\Lambda_i}L(W)$, $i = 0, 1$, of the tangent spaces to two distinct points Λ_0, Λ_1 of the Grassmannian. The pairing $\langle \dot\Lambda(t) \mid \dot\Lambda(\tau)\rangle$ of the velocities of the curve $t \mapsto \Lambda(t)$ gives a symmetric function of two variables which keeps all essential information about the curve. This function is defined out of the diagonal $\{t = \tau\}$ and has a very simple singularity at the diagonal:

$$\langle \dot\Lambda(t) \mid \dot\Lambda(\tau)\rangle = -\frac{k}{(t-\tau)^2} - g_{\Lambda}(t,\tau),$$

where k is the weight of the curve and $g_{\Lambda}(t,\tau)$ is a smooth function!

The first coming invariant of the parametrized curve, the generalized Ricci curvature, is just $g_{\Lambda}(t,t)$, the value of g_{Λ} at the diagonal. For regular curves and for rank 1 curves Ricci curvature is equal also to the trace of the defined earlier curvature operator.

In section 5 we are focused on unparametrized curves. Our investigation is based on a simple chain rule for a function g_{Λ}. Indeed, let $\varphi : \mathbb{R} \mapsto \mathbb{R}$ be a smooth monotonic function. It follows directly from definition of g_{Λ} that

$$g_{\Lambda\circ\varphi}(t,\tau) = \dot\varphi(t)\dot\varphi(\tau)g_{\Lambda}(\varphi(t),\varphi(\tau)) + k\left(\frac{\dot\varphi(t)\dot\varphi(\tau)}{(\varphi(t)-\varphi(\tau))^2} - \frac{1}{(t-\tau)^2}\right)$$

In particular,

$$g_{\Lambda\circ\varphi}(t,t) = \dot\varphi(t)^2 g_{\Lambda}(\varphi(t),\varphi(t)) + \frac{k}{3}\mathbb{S}(\varphi),$$

where $\mathbb{S}(\varphi)$ is a Schwarzian derivative of φ. The class of local parametrizations that kill the generalized Ricci curvature defines a canonical projective structure on the curve. The principal invariant of the unparametrized curve, the *fundamental form*, is a degree four differential on the curve; in the canonical projective parameter the fundamental form has the expression $\frac{1}{2}\frac{\partial^2 g_\Lambda}{\partial\tau^2}(t,t)(dt)^4$.

In section 6 we start a systematic study of the rank 1 curves and show that a rank 1 curve has a constant weight out of a discrete set of its interval of definition. In section 7 we prove that functions $\frac{\partial^{2i} g_\Lambda}{\partial \tau^{2i}}(t,t)$, $0 \le i \le m-1$ form a complete system of symplectic invariants of a rank 1 and a constant weight curve $\Lambda(\cdot)$ in the Lagrange Grassmannian $L(\mathbb{R}^{2m})$.

The Lagrange Grassmannian $L(\mathbb{R}^{2m})$ is a submanifold of the manifold $G(m, 2m)$ of all m-dimensional subspaces of $\mathbb{R}^{2m}$. The constructions of the derivative curve, the function g_Λ, the canonical projective structure, and the fundamental form can be done in the same way for general curves in $G(m, 2m)$.

2 Derivative Curve

From now on W will be the $2m$-dimensional linear space provided with the symplectic form $\bar{\sigma}$. Let Λ be a Lagrangian subspace of W, i.e. $\Lambda \in L(W)$. For any $w \in \Lambda$, the linear form $\bar{\sigma}(\cdot, w)$ vanishes on Λ and thus defines a linear form on W/Λ. The nondegeneracy of $\bar{\sigma}$ implies that the relation $w \mapsto \bar{\sigma}(\cdot, w)$, $w \in \Lambda$, induces a canonical isomorphism $\Lambda \cong (W/\Lambda)^*$ and, by the conjugation, $\Lambda^* \cong W/\Lambda$.

We set $\Lambda^\pitchfork = \{\Gamma \in L(W) : \Gamma \cap \Lambda = 0\}$, an open everywhere dense subset of $L(W)$. Let $Sym^2(\Lambda)$ be the space of self-adjoint linear mappings from Λ^* to Λ; this notation reflects the fact that $Sym^2(\Lambda)$ is the space of quadratic forms on Λ^* that is the symmetric square of Λ. $\Lambda^\pitchfork$ possesses a canonical structure of an affine space over the linear space $Sym^2(\Lambda) = Sym^2((W/\Lambda)^*)$. Indeed, for any $\Delta \in \Lambda^\pitchfork$ and coset $(w + \Lambda) \in W/\Lambda$, the intersection $\Delta \cap (w + \Lambda)$ of the linear subspace Δ and the affine subspace $w + \Lambda$ in W consists of exactly one point. To a pair $\Gamma, \Delta \in \Lambda^\pitchfork$ there corresponds a mapping $(\Gamma - \Delta) : W/\Lambda \to \Lambda$, where

$$(\Gamma - \Delta)(w + \Lambda) \stackrel{def}{=} \Gamma \cap (w + \Lambda) - \Delta \cap (w + \Lambda).$$

It is easy to check that the identification $W/\Lambda = \Lambda^*$ makes $(\Gamma - \Delta)$ a self-adjoint mapping from Λ^* to Λ. Moreover, given $\Delta \in \Lambda^\pitchfork$, the correspondence $\Gamma \mapsto (\Gamma - \Delta)$ is a one-to-one mapping of $\Lambda^\pitchfork$ onto $Sym^2(\Lambda)$ and the axioms of the affine space are obviously satisfied.

Fixing $\Delta \in \Lambda^\pitchfork$ one obtains a canonical identification $\Delta \cong W/\Lambda = \Lambda^*$. In particular, $(\Gamma - \Delta) \in Sym^2(\Lambda)$ turns into the mapping from Δ to Λ. For the last linear mapping we will use the notation $\langle \Delta, \Gamma, \Lambda \rangle : \Delta \to \Lambda$. In fact, this mapping has a much more straightforward description. Namely, the relations $W = \Delta \oplus \Lambda$, $\Gamma \cap \Lambda = 0$, imply that Γ is the graph of a linear mapping from Δ to Λ. Actually, it is the graph of the mapping $\langle \Delta, \Gamma, \Lambda \rangle$. In particular, $\ker\langle \Delta, \Gamma, \Lambda \rangle = \Delta \cap \Gamma$. If $\Delta \cap \Gamma = 0$, then $\langle \Lambda, \Gamma, \Delta \rangle = \langle \Delta, \Gamma, \Lambda \rangle^{-1}$.

Let us give coordinate representations of the introduced objects. We may assume that

$$W = \mathbb{R}^m \oplus \mathbb{R}^m = \{(x, y) : x, y \in \mathbb{R}^m\},$$

$$\bar{\sigma}((x_1, y_1), (x_2, y_2)) = \langle x_1, y_2 \rangle - \langle x_2, y_1 \rangle, \ \Lambda = \mathbb{R}^m \oplus 0, \ \Delta = 0 \oplus \mathbb{R}^m.$$

Then any $\Gamma \in \Delta^{\pitchfork}$ takes the form $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$, where S is a symmetric $m \times m$ matrix. The operator $\langle \Lambda, \Gamma, \Delta \rangle : \Lambda \to \Delta$ is represented by the matrix S, while the operator $\langle \Delta, \Gamma, \Lambda \rangle$ is represented by the matrix S^{-1}.

The coordinates in Λ induce the identification of $Sym^2\Lambda$ with the space of symmetric $m \times m$ matrices. $\Lambda^{\pitchfork}$ is an affine subspace over $Sym^2\Lambda$; we fix Δ as the origin in this affine subspace and thus obtain a coordinatization of $\Lambda^{\pitchfork}$ by symmetric $m \times m$ matrices. In particular, the "point" $\Gamma = \{(x, Sx) : x \in \mathbb{R}^n\}$ in $\Lambda^{\pitchfork}$ is represented by the matrix S^{-1}.

A subspace $\Gamma_0 = \{(x, S_0 x) : x \in \mathbb{R}^n\}$ is transversal to Γ if and only if $det(S - S_0) \neq 0$. Let us pick coordinates $\{x\}$ in Γ_0 and fix Δ as the origin in the affine space $\Gamma_0^{\pitchfork}$. In the induced coordinatization of $\Gamma_0^{\pitchfork}$ the "point" Γ is represented by the matrix $(S - S_0)^{-1}$.

Let $t \mapsto \Lambda(t)$ be a smooth curve in $L(W)$ defined on some interval $I \subset \mathbb{R}$. We say that the curve $\Lambda(\cdot)$ is *ample at* τ if $\exists s > 0$ such that for any representative $\Lambda_\tau^s(\cdot)$ of the s-jet of $\Lambda(\cdot)$ at τ, $\exists t$ such that $\Lambda_\tau^s(t) \cap \Lambda(\tau) = 0$. The curve $\Lambda(\cdot)$ is called *ample* if it is ample at any point.

We have given an intrinsic definition of an ample curve. In coordinates it takes the following form: the curve $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ is ample at τ if and only if the function $t \mapsto det(S_t - S_\tau)$ has a root of finite order at τ.

The following lemma shows that analytic monotonic curve (monotonic means that the curve has nonnegative or nonpositive velocities at any point) can be actually reduced to the ample curve by an appropriate factorization.

Lemma 2.1 *Let $\Lambda(t)$ be analytic monotonic curve in $L(W)$. Then for any parameter τ there exists a subspace K of $\Lambda(\tau)$ such that for all t sufficiently closed to τ the following holds*

$$K = \Lambda(t) \cap \Lambda(\tau). \tag{2.1}$$

In addition, if $\Lambda(t)$ is not a constant curve, then the curve $t \mapsto \Lambda(t)/K$ is a well defined ample curve in the Lagrange Grassmannian $L(K^\angle/K)$.

Proof. Without loss of generality suppose that the curve $\Lambda(t)$ is nondecreasing (i.e., has nonnegative definite velocity at any point). Denote $K_t = \Lambda(t) \cap \Lambda(\tau)$. Let $t \mapsto \{(x, S_t x) : x \in \mathbb{R}^n\}$ be coordinate representation of the germ of $\Lambda(t)$ at τ. Then $K_t = Ker(S_t - S_\tau)$. By assumption $v \to \langle \frac{d}{dt} S_t v, v \rangle$ is nonnegative definite quadratic form on $\Lambda(\tau)$. It implies that $K_{t_2} \subset K_{t_1}$ for $t < t_1 < t_2$. Therefore for $t > \tau$ sufficiently closed to τ the subspace K_t does not depend on t and will be denoted by K. By analyticity the subspaces $K \subset \Lambda(t)$ for any t and the curve $t \mapsto \Lambda(t)/K$ is well defined ample curve in the Lagrange Grassmannian $L(K^\angle/K)$. $\square$.

Assume that $\Lambda(\cdot)$ is ample at τ. Then $\Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ for all t from a punctured neighborhood of τ. We obtain the curve $t \mapsto \Lambda(t) \in \Lambda(\tau)^{\pitchfork}$ in the affine space $\Lambda(\tau)^{\pitchfork}$ with the pole at τ. We denote by $\Lambda_\tau(t)$ the identical imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The subscript τ is not superfluous, since the affine structure depends on $\Lambda(\tau)$ and then on τ. Fixing an "origin" in $\Lambda(\tau)^{\pitchfork}$ we make $\Lambda_\tau(t)$ a vector function with values in $Sym^2(\Lambda)$ and with the pole at $t = \tau$. Such a vector function admits the expansion in the Laurent series at τ. Obviously, only free term in the Laurent expansion depends on the choice of the "origin" we did to identify the affine space with the linear one. More precisely, the addition of a vector to the "origin" results in the addition of the same vector to the free term in the Laurent expansion. In other words, for the Laurent expansion of a curve in an affine space, the free term of the expansion is a point of this affine space while all other terms are elements of the corresponding linear space. In particular,

$$\Lambda_\tau(t) \approx \Lambda^0(\tau) + \sum_{\substack{i=-l \\ i \neq 0}}^{\infty} Q_i(\tau)(t-\tau)^i, \tag{2.2}$$

where $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$, $Q_i(\tau) \in Sym^2\Lambda(\tau)$.

Assume that the curve $\Lambda(\cdot)$ is ample. Then $\Lambda^0(\tau) \in \Lambda(\tau)^{\pitchfork}$ is defined for all τ. The curve $\tau \mapsto \Lambda^0(\tau)$ is called the *derivative curve* of $\Lambda(\cdot)$.

Another characterization of $\Lambda^0(\tau)$ can be done in terms of the curves $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau)\rangle$ in the linear space $Hom(\Delta, \Lambda(\tau))$, $\Delta \in \Lambda(\tau)^{\pitchfork}$. These curves have poles at τ. The Laurent expansion at $t = \tau$ of the vector function $t \mapsto \langle \Delta, \Lambda(t), \Lambda(\tau)\rangle$ has zero free term if and only if $\Delta = \Lambda^0(\tau)$.

The coordinate version of the series (2.2) is the Laurent expansion of the matrix-valued function $t \mapsto (S_t - S_\tau)^{-1}$ at $t = \tau$, where $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$.

Suppose that

$$(S_t - S_\tau)^{-1} \approx \sum_{i=-l}^{\infty} A_i(\tau)(t-\tau)^i, \tag{2.3}$$

Differentiating both sides of (2.3) w.r.t τ and comparing coefficients of the corresponding expansions one can get the following recursive type formula for the coefficients $A_i(\tau)$

$$\frac{d}{d\tau}A_i(\tau) = (i+1)A_{i+1}(\tau) + \sum_{j=-l}^{i+l} A_j(\tau)\dot{S}_\tau A_{i-j}(\tau) \tag{2.4}$$

that will be used in the sequel.

For monotonic ample curve $\Lambda : I \subset \mathbb{R} \mapsto L(W)$ we introduce the following two notions

Definition 1 *The rank of the velocity $\dot{\Lambda}(\tau)$ will be called a rank of the curve $\Lambda(\cdot)$ at τ. The order of zero of the function $t \mapsto det(S_t - S_\tau)$ at τ, where S_t is a coordinate representation of $\Lambda(\cdot)$, is called a weight of $\Lambda(\cdot)$ at τ.*

It is easy to see that the rank and the weight of $\Lambda(\tau)$ are integral valued upper semicontinuous functions of τ. In particular, they are locally constant on the open dense subset of the interval of definition I. In the sequel we will be mostly concentrated on the monotonic ample curves of the constant rank and weight.

3 Curvature operator and regular curves.

Using derivative curve one can construct an operator invariant of the curve $\Lambda(t)$ at any its point. Namely, take velocities $\dot{\Lambda}(t)$ and $\dot{\Lambda}^0(t)$ of $\Lambda(t)$ and its derivative curve $\Lambda^0(t)$. Note that $\dot{\Lambda}(t)$ is linear operator from $\Lambda(t)$ to $\Lambda(t)^*$ and $\dot{\Lambda}^0(t)$ is linear operator from $\Lambda^0(t)$ to $\Lambda^0(t)^*$. Since the form σ defines the canonical isomorphism between $\Lambda^0(t)$ and $\Lambda(t)^*$, the following operator $R(t) : \Lambda(t) \to \Lambda(t)$ can be defined:

$$R(t) = -\dot{\Lambda}^0(t) \circ \dot{\Lambda}(t) \tag{3.1}$$

This operator is called *curvature operator* of Λ at t.

Remark 1 *In the case of Riemannian geometry the operator $R(t)$ is similar to the so-called Ricci operator $v \to R^\nabla(\dot{\gamma}(t), v)\dot{\gamma}(t)$, which appears in the classical Jacobi equation $\nabla_{\dot{\gamma}(t)}\nabla_{\dot{\gamma}(t)}V + R^\nabla(\dot{\gamma}(t), V)\dot{\gamma}(t) = 0$ for Jacobi vector fields V along the geodesic $\gamma(t)$ (here R^∇ is curvature tensor of Levi-Civita connection ∇), see [1]. This is the reason for the sign "$-$" in (3.1).*

The curvature operator plays an important role for so-called regular curves. The curve $\Lambda(t)$ in Lagrangian Grassmannian is called *regular*, if the quadratic form $\dot{\Lambda}(t)$ is nondegenerated for all t. If the curve $\Lambda(\cdot)$ is regular and has a coordinate representation $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$ then the function $t \mapsto (S_t - S_\tau)^{-1}$ has a simple pole at τ. Indeed,

$$(S_t - S_\tau)^{-1} = \Big(\dot{S}_\tau(t-\tau) + O((t-\tau)^2)\Big)^{-1} = \frac{\dot{S}_\tau^{-1}}{t-\tau}\Big(I + O(t-\tau)\Big)^{-1} = \frac{\dot{S}_\tau^{-1}}{t-\tau} + O(1) \tag{3.2}$$

So, in the notation of (2.3) for the regular curve we have $l = 1$, $A_{-1} = \dot{S}_\tau^{-1}$ and relation (2.4) can be transformed into the following recursive formula

$$A_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}A_i(\tau) - \sum_{j=0}^{i} A_j(\tau)\dot{S}_\tau A_{i-j}(\tau)\right) \tag{3.3}$$

In particular,

$$A_0(\tau)=\frac{1}{2}\frac{d}{d\tau}A_{-1}(\tau)=-\frac{1}{2}\dot S_\tau^{-1}\ddot S_\tau\dot S_\tau^{-1} \tag{3.4}$$

and, by direct calculations,

$$A_1(\tau)=\frac{1}{3}(\frac{d}{d\tau}A_0(\tau)-A_0(\tau)\dot S_\tau A_0(\tau))=\left(\frac{1}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2-\frac{1}{6}\dot S_\tau^{-1}S_\tau^{(3)}\right)\dot S_\tau^{-1} \tag{3.5}$$

For a given τ one can choose a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\tau)=0$. Namely take S_t the matrix of the linear mapping $<\Lambda(\tau),\Lambda(t),\Lambda^0(\tau)>$. In this coordinate representation the derivative $\dot A_0(\tau)$ is a matrix corresponding to the velocity $\dot\Lambda^0(\tau)$ of the derivative curve. Also, from (3.5) it follows that $\dot A_0(\tau)=3A_1(\tau)$. This together with (3.1) implies that the matrix $\mathcal{R}(\tau)$ corresponding in the chosen basis of $\Lambda(\tau)$ to the curvature operator $R(\tau)$ has the following form

$$\mathcal{R}(\tau)=-3A_1(\tau)(A_{-1}(\tau))^{-1}=\frac{1}{2}\dot S_\tau^{-1}S_\tau^{(3)}-\frac{3}{4}\left(\dot S_\tau^{-1}\ddot S_\tau\right)^2=$$

$$\frac{d}{d\tau}\left((2\dot S_\tau)^{-1}\ddot S_\tau\right)-\left((2\dot S_\tau)^{-1}\ddot S_\tau\right)^2 \tag{3.6}$$

Since $Q_1(\tau)\circ(Q_{-1}(\tau))^{-1}:\Lambda(\tau)\mapsto\Lambda(\tau)$ is well defined operator, we can write the first equality of (3.6) in the following operator form

$$R(\tau)=Q_1(\tau)\circ(Q_{-1}(\tau))^{-1} \tag{3.7}$$

This implies actually that the formula (3.6) is also true for any coordinate representation S_t of the curve $\Lambda(t)$ (even without the assumption that $A_0(\tau)=0$).

Note that the right-hand side of (3.6) is a matrix analog of so-called Schwarz derivative or Schwarzian. Let us recall that the differential operator:

$$\mathbb{S}:\varphi\mapsto\frac{1}{2}\frac{\varphi^{(3)}}{\varphi'}-\frac{3}{4}\Big(\frac{\varphi''}{\varphi'}\Big)^2=\frac{d}{dt}\Big(\frac{\varphi''}{2\varphi'}\Big)-\Big(\frac{\varphi''}{2\varphi'}\Big)^2, \tag{3.8}$$

acting on scalar function φ is called *Schwarzian.* The operator $\mathbb{S}$ is characterized by the following remarkable property: General solution of the equation $\mathbb{S}\varphi=\rho$ w.r.t φ is a Möbius transformation (with constant coefficients) of some particular solution of this equation. The matrix analog of this operator has similar property, concerning "matrix Möbius transformation" of the type $S\mapsto (C+DS)(A+BS)^{-1}$. In particular, if $R(t)\equiv 0$, then the coordinate representation S_t of our curve has the form

$$S_t=(C+Dt)(A+Bt)^{-1}$$

where

$$\begin{pmatrix} A & B\\ C & D\end{pmatrix}\in Sp(2m)$$

For further information about the regular curves we refer to [1].

4 Expansion of the cross-ratio and Ricci curvature.

For the nonregular curve $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^n\}$, the function $t \mapsto (S_t - S_\tau)^{-1}$ has a pole of order greater than 1 at τ and it is much more difficult to compute its Laurent expansion. For example, in the nonregular case there is no direct recursive formula like (3.3). In this section we show how to construct numerical invariants for curves with constant weight using the notion of cross-ratio of four "points" in the Lagrange Grassmannian.

Let Λ_0, Λ_1, Λ_2, and Λ_3 be Lagrangian subspaces of W such that $\Lambda_0 \cap \Lambda_3 = \Lambda_1 \cap \Lambda_2 = 0$. Also suppose for simplicity that $\Lambda_0 \cap \Lambda_2 = 0$. The following linear mappings $\langle \Lambda_0, \Lambda_1, \Lambda_2 \rangle : \Lambda_0 \to \Lambda_2$, $\langle \Lambda_2, \Lambda_3, \Lambda_0 \rangle : \Lambda_2 \to \Lambda_0$ are well defined. The cross-ratio $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ of four "points" Λ_0, Λ_1, Λ_2, and Λ_3 in the Lagrangian Grassmannian is, by definition, the following linear operator in Λ_0:

$$\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big] = \langle \Lambda_2, \Lambda_3, \Lambda_0 \rangle \langle \Lambda_0, \Lambda_1, \Lambda_2 \rangle. \tag{4.1}$$

This notion is a "matrix" analog of the classical cross-ratio of four points in the projective line. Indeed, let $\Lambda_i = \{(x, S_i x) : x \in \mathbb{R}^n\}$, then, in coordinates $\{x\}$, the cross-ratio takes the form:

$$\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big] = (S_0 - S_3)^{-1}(S_3 - S_2)(S_2 - S_1)^{-1}(S_1 - S_0) \tag{4.2}$$

By construction, all coefficients of the characteristic polynomial of $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ are invariants of four subspaces $\Lambda_0, \Lambda_1, \Lambda_2$, and Λ_3.

The assumption that $\Lambda_0 \cap \Lambda_2 = 0$ is satisfied in our further considerations but the cross-ratio can be defined also without this assumption. Indeed, the matrix in the righthand side of (4.2) is well defined also in the case $\Lambda_0 \cap \Lambda_2 \neq 0$ and this matrix is transformed to a similar matrix under any change of coordinates. So, we obtain the class of similar matrices that is symplectic invariant of four subspaces $\Lambda_0, \Lambda_1, \Lambda_2$, and Λ_3. This class can be taken as a definition of cross-ratio $\Big[\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3\Big]$ (see [7] for the details).

Given two tangent vectors $V_0 \in T_{\Lambda_0} L(W)$ and $V_1 \in T_{\Lambda_1} L(W)$, where Λ_0 and Λ_1 are transversal Lagrangian subspaces, one can define an infinitesimal analog of the cross-ratio. V_0 is the self-adjoint linear mapping from Λ_0 to Λ_0^*. The form σ identifies canonically Λ_0^* with Λ_1. Under this identification V_0 can be considered as the linear mapping from Λ_0 to Λ_1. In the same way, identifying Λ_1^* with Λ_0, we look on V_1 as on the operator from Λ_1 to Λ_0. Therefore, the following operator $V_1 \odot V_0 : \Lambda_0 \to \Lambda_0$ can be defined

$$V_1 \odot V_0 \overset{def}{=} V_1 \circ V_0 \tag{4.3}$$

This operator will be called an *infinitesimal cross-ratio* of a pair $(V_0, V_1) \in T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$. The infinitesimal cross-ratio is symplectic invariant of the tangent vectors V_0 and V_1.

One can define the following bilinear form $\langle \cdot \mid \cdot \rangle_{\Lambda_0,\Lambda_1}$ on $T_{\Lambda_0}L(W) \times T_{\Lambda_1}L(W)$:

$$\langle V_0 \mid V_1 \rangle_{\Lambda_0,\Lambda_1} \stackrel{def}{=} \operatorname{tr}(V_0 \odot V_1) \tag{4.4}$$

This bilinear form will be called an *inner pairing* of the tangent spaces $T_{\Lambda_0}L(W)$ and $T_{\Lambda_1}L(W)$.

If $\Lambda_i = \{(x, S_i x) : x \in \mathbb{R}^n\}$ and P_i are symmetric matrices corresponding to V_i, $i = 0, 1$, then

$$V_1 \odot V_0 = (S_0 - S_1)^{-1} P_1 (S_1 - S_0)^{-1} P_0 \tag{4.5}$$

First note that if the curve $\Lambda(t)$ is regular, then for any t_0 it is easy to expand the following operator function

$$(t_1, t_2, t_3) \mapsto \frac{\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]} \tag{4.6}$$

in the Taylor expansion at the diagonal point (t_0, t_0, t_0), where $\Big[t_0, t_1, t_2, t_3\Big] = \frac{(t_1 - t_0)(t_3 - t_2)}{(t_2 - t_1)(t_0 - t_3)}$ is the usual cross-ratio of four numbers t_0, t_1, t_2, and t_3. Namely, the following expansion

$$\frac{\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]} = I + \frac{1}{3} R(t_0)(t_2 - t_0)(t_3 - t_1) + O\left(\left(\sum_{i=1}^{3} (t_i - t_0)^2\right)^{3/2}\right) \tag{4.7}$$

is valid, where, as before, $R(t)$ is the curvature operator. Relation (4.7) shows that the curvature operator is the first nontrivial coefficient of the Taylor expansion of the cross-ratio.

Unfortunately, for the nonregular curves there are no simple expansions of the operator function (4.6) or any other operator functions, involving cross-ratio itself. Instead of this one can try to expand the coefficients of the characteristic polynomial of the cross-ratio. Now we are going to show how to use this idea to construct invariants of the curve $\Lambda(t)$ of the constant weight k in $L(W)$.

By the above the function $(t_0, t_1, t_2, t_3) \to \det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]$ is symplectic invariants of the curve $\Lambda(t)$. Using this fact, let us try to find symplectic invariants of $\Lambda(t)$ that are functions of t. First we introduce the following function:

$$\mathcal{G}(t_0, t_1, t_2, t_3) = \ln\left(\frac{det\Big[\Lambda(t_0), \Lambda(t_1), \Lambda(t_2), \Lambda(t_3)\Big]}{\Big[t_0, t_1, t_2, t_3\Big]^k}\right), \tag{4.8}$$

The function $\mathcal{G}(t_0,t_1,t_2,t_3)$ is also a symplectic invariant of $\Lambda(t)$ and in addition it can be defined as a smooth function in a neighborhood of any diagonal point (t,t,t,t) . Indeed, by the definition of weight

$$det(S_{t_0}-S_{t_1})=(t_0-t_1)^k X(t_0,t_1), \tag{4.9}$$

where

$$X(t,t)\neq 0 \tag{4.10}$$

for any t. The function $X(t_0,t_1)$ is symmetric, since by changing the order in (4.9) we obtain that X can be symmetric or antisymmetric, but the last case is impossible by (4.10).

Let us define another symmetric function

$$f(t_0,t_1)=lnX(t_0,t_1) \tag{4.11}$$

The function $f(t_0,t_1)$ is smooth in a neighborhood of any diagonal point (t,t) and by (4.2), (4.8)

$$\mathcal{G}(t_0,t_1,t_2,t_3)=f(t_1,t_0)-f(t_2,t_1)+f(t_3,t_2)-f(t_0,t_3) \tag{4.12}$$

Hence $\mathcal{G}(t_0,t_1,t_2,t_3)$ can be defined as a smooth function in a neighborhood of any diagonal point (t,t,t,t). Using this fact one can construct the following functions of two variables that are symplectic invariants of the curve $\Lambda(t)$

$$h(t_0,t_1)=\mathcal{G}(t_0,t_1,t_1,t_0)=2f(t_0,t_1)-f(t_0,t_0)-f(t_1,t_1) \tag{4.13}$$

$$g(t_0,t_1)=\frac{1}{2}\frac{\partial^2}{\partial t_0\partial t_1}h(t_0,t_1)=\frac{\partial^2}{\partial t_0\partial t_1}f(t_0,t_1) \tag{4.14}$$

On the contrary the function $f(t_0,t_1)$ depends on the choice of the coordinate representation S_t.

It follows from (4.13) that $h(t_0,t_0)\equiv 0$ and $\frac{\partial}{\partial t_0}h(t_0,t_0)\equiv 0$. Therefore the function $h(t_0,t_1)$ can be recovered from $g(t_0,t_1)$. Moreover, the function $\mathcal{G}(t_0,t_1,t_2,t_3)$ can be easily recovered from $h(t_0,t_1)$ (and therefore from $g(t_0,t_1)$). Namely, by (4.12) and (4.14)

$$\mathcal{G}(t_0,t_1,t_2,t_3)=\frac{1}{2}\left(h(t_1,t_0)-h(t_2,t_1)+h(t_3,t_2)-h(t_0,t_3)\right) \tag{4.15}$$

So, g or h keep all the information on $\mathcal{G}$ and thus on $\det\Big[\Lambda(t_0),\Lambda(t_1),\Lambda(t_2),\Lambda(t_3)\Big]$.

The function $g(t_0,t_1)$ can be expanded in the formal Taylor series at the point (t,t) in the following way:

$$g(t_0,t_1) \approx \sum_{i,j=0}^{\infty} \beta_{i,j}(t)(t_0-t)^i(t_1-t)^j \tag{4.16}$$

with

$$\beta_{i,j}(t) = \beta_{j,i}(t) \tag{4.17}$$

Since the function g is symplectic invariant of the curve $\Lambda(t)$, all coefficients $\beta_{i,j}(t)$, $i,j \geq 0$, are also symplectic invariants.

The following natural questions arises: Does the function $g(t_0,t_1)$ determine the curve $\Lambda(t)$ with prescribed rank and weight uniquely, up to a symplectic transformation, and which set of the coefficients $\beta_{i,j}(t)$ determine the function $g(t_0,t_1)$? We shall give the positive answers on both of these questions in the section 7 for the curve of rank 1 (see Theorems 1 and 2).

Meanwhile, let us prove the following simple relation between coefficients $\beta_{i,j}(t)$

$$\beta'_{i,j}(t) = (i+1)\beta_{i+1,j} + (j+1)\beta_{i,j+1} \tag{4.18}$$

Indeed, from (4.16) it follows that

$$\beta_{i,j}(t) = \frac{1}{i!\,j!}\frac{\partial^{i+j} g}{\partial t_0^i \partial t_1^j}(t,t).$$

Therefore

$$\beta'_{i,j}(t) = \frac{1}{i!\,j!}\Big(\frac{\partial^{i+j+1}}{\partial t_0^{i+1}\partial t_1^j} g(t,t) + \frac{\partial^{i+j+1}}{\partial t_0^{i}\partial t_1^{j+1}} g(t,t)\Big) = \frac{1}{i!\,j!}\Big((i+1)!\,j!\,\beta_{i+1,j}(t) + i!\,(j+1)!\,\beta_{i,j+1}(t)\Big)$$

that implies (4.18).

As a corollary of the relation (4.18) we obtain the following lemma

Lemma 4.1 *The coefficients $\beta_{0,2k}(t)$, $k \geq 0$ determine uniquely the formal expansion (4.16).*

Proof. For a given $n \geq 0$ let us consider all equations of the type (4.18) with $i+j=n$ and $i \leq j$. Consider two cases

1) If n is even then we have $\frac{n}{2}+1$ independent equations with respect to $\frac{n}{2}+1$ variables $\beta_{i,j}(t)$, $i+j=n+1$, $0 \leq i < \frac{n}{2}$. This fact together with symmetric relation (4.17) implies that all $\beta_{i,j}(t)$ with $i+j=n+1$ can be expressed by derivatives of $\beta_{i,j}$ with $i+j=n$.

2) If n is odd then we have $\frac{n+1}{2}$ independent equations with respect to $\frac{n+1}{2}+1$ variables $\beta_{i,j}$, $i+j=n+1$, $0 \leq i < \frac{n+1}{2}$. Starting from $i=0$ one can express step by step all $\beta_{i,j}$, $i+j=n+1$,

$1 \le i < \frac{n+1}{2}$ by $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i+j=n$. Then by symmetric relation (4.17) we have that all coefficients $\beta_{i,j}(t)$ with $i+j=n+1$ can be expressed by $\beta_{0,n+1}$ and derivatives of $\beta_{i,j}$ with $i+j=n$.

So, starting from $n=0$ and applying step by step the arguments of 1) and 2), one can expressed all $\beta_{i,j}(t)$ by $\beta_{0,2k}(t)$, $k \ge 0$, and their derivatives. $\square$

It turns out that there is simple connection between function g, the inner pairing defined by (4.4), and the coefficients Q_i of the Laurent expansion (2.2).

Lemma 4.2 *The following relations hold*

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = -\frac{k}{(t-\tau)^2} - g(t,\tau) \tag{4.19}$$

$$\operatorname{tr}\left(Q_i(t)\dot\Lambda(t)\right) = 0,\ i < -1, \tag{4.20}$$

$$\operatorname{tr}\left(Q_{-1}(t)\dot\Lambda(t)\right) = k, \tag{4.21}$$

$$\operatorname{tr}\left(Q_i(t)\dot\Lambda(t)\right) = -\frac{1}{i}\beta_{0,i-1}(t),\ i \in \mathbb{N} \tag{4.22}$$

Proof. Let $\Lambda_\tau(t)$ be the identical imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$ (see section 2). Then the inner pairing $\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)}$ can be expressed in the following way

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t)\circ\dot\Lambda(\tau)\right) \tag{4.23}$$

In the coordinates the previous relation can be written as follows

$$\langle\, \dot\Lambda(t) \mid \dot\Lambda(\tau)\,\rangle_{\Lambda(t),\Lambda(\tau)} = \operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\ \dot S_\tau\right) \tag{4.24}$$

Let us prove (4.19). By definition

$$\ln(\det(S_t - S_\tau)) = k\ln(t-\tau) + f(t,\tau)$$

Differentiating the last equality w.r.t. τ and using the fact that $\frac{d}{d\tau}\Big(\ln(\det Y(\tau))\Big) = \operatorname{tr}\Big((Y(\tau))^{-1}\dot Y(\tau)\Big)$ for some matrix curve $Y(\tau)$ we obtain:

$$-\mathrm{tr}\Big((S_t - S_\tau)^{-1}\dot{S}_\tau\Big) = -\frac{k}{t-\tau} + \frac{\partial}{\partial\tau}f(t,\tau)$$

Differentiating the last equality w.r.t. t and using (4.14) we get

$$-\mathrm{tr}\Big(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\dot{S}_\tau\Big) = \frac{k}{(t-\tau)^2} + \frac{\partial^2}{\partial t\partial\tau}f(t,\tau) = \frac{k}{(t-\tau)^2} + g(t,\tau)$$

This together with (4.24) implies (4.19).

In order to prove (4.20)-(4.22) let us expand both sides of (4.19) in the correspponding formal series. On one hand by (2.2) we have

$$\mathrm{tr}\Big(\frac{\partial}{\partial t}\Lambda_\tau(t)\circ\dot{\Lambda}(\tau)\Big) \approx \sum_{i=-l-1}^{\infty}(i+1)\mathrm{tr}\Big(Q_{i+1}(\tau)\dot{\Lambda}(\tau)\Big)(t-\tau)^i \tag{4.25}$$

On the other hand by (4.16)

$$g(t,\tau) \approx \sum_{0}^{\infty}\beta_{0,i}(\tau)(t-\tau)^i \tag{4.26}$$

Comparing coefficients of (4.25) and (4.26) we get (4.20)-(4.22). □

For the regular curve using (3.6) and applying formula (4.22) to the first appearing in (4.16) coefficient $\beta_{0,0}(t)$ we obtain

$$\beta_{0,0}(t) = \frac{1}{3}\,\mathrm{tr}\,R(t) = \frac{1}{3}\,\mathrm{tr}\,\mathbb{S}(S_t), \tag{4.27}$$

where $\mathbb{S}$ denotes Schwarz operator. The last relation and Remark 1 shows that $\beta_{0,0}$ generalizes the Ricci curvature in the Riemannian geometry. It justifies the following definition for the general curve of constant rank and weight

Definition 2 *The first appearing in (4.16) coefficient $\beta_{0,0}(t)$ is called Ricci curvature of $\Lambda(t)$.*

In the sequel the Ricci curvature will be denoted by $\rho(t)$.

At the end of this section we compute the expansion of $g(t_0,t_1)$ in the case $\dim W = 2$. In this case $L(W)$ is in fact the real projective line $\mathbb{RP}^1$ and coordinate representation S_t of the curve is scalar function . Therefore the relation (4.22) can be rewritten in the following form

$$\beta_{0,i} = -(i+1)A_{i+1}(t)\dot{S}_t,$$

where A_i are as in (2.3). In particular from (4.27) it follows that

$$\rho(t) = \frac{1}{3}\mathbb{S}(S_t), \tag{4.28}$$

i.e., in the scalar case the Ricci curvature of the curve $\Lambda(t)$ is Schwarzian of its coordinate representation.

Denote by

$$B_i(\tau) = -\frac{1}{i}\beta_{0.i-1} = A(\tau)\dot{S}_\tau \tag{4.29}$$

Multiplying both sides of (3.3) by $\dot{S}_\tau$ and using commutativity of multiplication in the scalar case, one can easily obtain the following recursive formula for $B_i(\tau)$.

$$B_{i+1}(\tau) = \frac{1}{i+3}\left(\frac{d}{d\tau}B_i(\tau) - \sum_{j=1}^{i-1} B_j(\tau)B_{i-j}(\tau)\right), \; i \in \mathbb{N} \tag{4.30}$$

As a consequence of Lemma 4.1 and formulas (4.28)– (4.30) one can obtain the following

Proposition 2 *In the scalar case (i.e., $dimW = 2$) all coefficients $\beta_{i,j}(t)$ can be expressed by Ricci curvature (that is Schwarzian of any coordinate representation of the curve $\Lambda(t)$) and its derivative. The function $g(t,\tau)$ is identically equal to zero iff coordinate representations of the curve $\Lambda(t)$ are Möbius transformations.*

5 Fundamental form of the unparametrized curve.

The Jacobi curve constructed in the Introduction is actually unparametrized curve, i.e., one-dimensional submanifolds in Lagrange Grassmannian. Therefore it is natural to find symplectic invariants of unparametrized curves in $L(W)$. Especially it is important for Jacobi curves of abnormal extremals which (in opposite to the normal extremals) a priori have no special parametrizations.

First of all we want to show how, using the Ricci curvature, one can define a canonical projective structure on the unparametrized curve $\Lambda(\cdot)$. For this let us check how the Ricci curvature is transformed by a reparametrization of the curve $\Lambda(t)$.

Let $\tau = \varphi(t)$ be a reparametrization and let $\bar{\Lambda}(\tau) = \Lambda(\varphi^{-1}(\tau))$. For some coordinate representation S_t of $\Lambda(t)$ let $\bar{S_\tau} = S_{\varphi^{-1}(\tau)}$ be the coordinate representation of $\bar{\Lambda}(\tau)$. Denote by $\bar{f}$ the function playing for $\bar{S}_\tau$ the same role as the function f defined by (4.11) plays for S_t. Then from (4.11) it follows that

$$\bar{f}(\tau_0,\tau_1) = f(t_0,t_1) - k\,\ln\left(\frac{\varphi(t_0)-\varphi(t_1)}{t_0-t_1}\right), \tag{5.1}$$

where $\tau_i = \varphi(t_i)$, $i = 0, 1$.

Now denote by $\bar{g}$, $\bar{\beta}_{i,j}$ functions playing for $\bar{\Lambda}(\tau)$ the same role as the functions g,$\beta_{i,j}$ defined by (4.14) and (4.16) play for $\Lambda(t)$.

Note also that we can look on the function $\varphi(t)$ as on the coordinate representation of some curve in $\mathbb{RP}^1 = L(W)$ with $\dim W = 2$. So, all constructions and formulas of the previous section can be applied to this case. We denote by $g_\varphi(t_0, t_1)$ the function defined by (4.11), (4.14) with S_t replaced by $\varphi(t)$. Then differentiating both sides of (5.1) once w.r.t. t_0 and twice t_1 we get

$$\bar{g}\left(\varphi(t_0), \varphi(t_1)\right) \varphi'(t_0)\varphi'(t_1) = g(t_0, t_1) - k g_\varphi(t_0, t_1), \tag{5.2}$$

By (4.16) and (4.28) it follows that the substitution $t_0 = t_1 = t$ into (5.2) give us the following reparametrization rule for Ricci curvature

$$\bar{\rho}(\tau)(\varphi'(t))^2 = \rho(t) - \frac{k}{3}\,\mathbb{S}(\varphi(t)) \tag{5.3}$$

Now we would like to find all reparametrizations $\tau = \varphi(t)$ such that the Ricci curvature $\bar{\rho}(\tau)$ in the new parameter τ is identically equal to zero. The reparametrization rule (5.3) implies that such reparametrization have to satisfy the following differential equation

$$\mathbb{S}\left(\varphi(t)\right) = \frac{3\rho(t)}{k} \tag{5.4}$$

This equation has a solutions at least locally (i.e., in a neighborhood of any given point) and as was mentioned already in the section 3 any two solution are transformed one to another by Möbius transformation. In other words the set of all parametrization of $\Lambda(\cdot)$ with Ricci curvature identically equal to zero defines a projective structure on $\Lambda(\cdot)$ (any two parametrization from this set are transformed one to another by Möbius transformation). We call it *the canonical projective structure* of the curve $\Lambda(\cdot)$. The parameters of the canonical projective structure will be called *projective parameters.*

Now we give a construction of a special form on unparametrized curve $\Lambda(\cdot)$ (namely, the differential of degree four on $\Lambda(\cdot)$), which is the first appearing invariant of the unparametrized curve. We will call it the fundamental form of the curve $\Lambda(\cdot)$.

Let t be a projective parameter on $\Lambda(\cdot)$. Then by definition $\rho(t) \equiv 0$, and by (4.18) $\beta_{0,1}(t) \equiv \frac{1}{2}\beta'_{0,0}(t) \equiv 0$. Therefore by (4.26) we obtain that in projective parameter

$$g(t_0, t_1) = \beta_{0,2}(t_0)(t_1 - t_0)^2 + O\left((t_1 - t_0)^3\right) \tag{5.5}$$

Let τ be another projective parameter on $\Lambda(\cdot)$ i.e., $\tau = \varphi(t) = \frac{at+b}{ct+d}$. Then by Proposition 2 $g_\varphi(t_0, t_1) \equiv 0$. Substituting this to the (5.2) we have

$$\bar{g}\left(\varphi(t_0), \varphi(t_1)\right) \varphi'(t_0)\varphi'(t_1) = g(t_0, t_1), \tag{5.6}$$

where $\tau_i = \varphi(t_i)$, $i = 0, 1$. Using (5.5), we compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result we obtain

$$\bar{\beta}_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0)$$

or

$$\bar{\beta}_{0,2}(\tau)(d\tau)^4 = \beta_{0,2}(t)(dt)^4 \tag{5.7}$$

It means that the form $\beta_{0,2}(t)(dt)^4$ does not depend on the choice of the projective parameter t. We will call this form *a fundamental form* of the curve $\Lambda(\cdot)$ and will denote by $\mathcal{A}$.

If t is an arbitrary (not necessarily projective) parameter on the curve $\Lambda(\cdot)$, then the fundamental form $\mathcal{A}$ in this parameter has to be of the form $A(t)(dt)^4$, where $A(t)$ is a smooth function (the "density" of the fundamental form).

Lemma 5.1 *For arbitrary parameter t the density $A(t)$ of the fundamental form satisfies the following relation*

$$A(t) = \beta_{0,2}(t) - \frac{3}{5k}\rho(t)^2 - \frac{3}{20}\rho''(t) \tag{5.8}$$

or, equivalently,

$$A(t) = \Big(\frac{1}{10}\left(\frac{\partial}{\partial t_0} + \frac{\partial}{\partial t_1}\right)^2 - \frac{1}{2}\frac{\partial^2}{\partial t_0 \partial t_1}\Big) g(t_0, t_1)\Big|_{t_0=t_1=t} - \frac{3}{5k} g(t,t)^2. \tag{5.9}$$

Proof. Let $\tau = \varphi(t)$ be a reparametrization such that τ be a projective parameter. It means that $\varphi(\tau)$ satisfies the equation (5.4). Denote by $\beta^{\varphi}_{i,j}(t_0, t_1)$ the coefficients defined by (4.11), (4.14) and (4.16) with S_t replaced by $\varphi(t)$. Using (5.5), compare the coefficients of the first terms in the Taylor expansions of both sides of (5.6). As a result we obtain

$$\bar{\beta}_{0,2}(\varphi(t_0))(\varphi'(t_0))^4 = \beta_{0,2}(t_0) - k\beta^{\varphi}_{0,2}(t_0)$$

or

$$\mathcal{A} = \bar{\beta}_{0,2}(\tau)(d\tau)^4 = \left(\beta_{0,2}(t) - k\beta^{\varphi}_{0,2}(t)\right)(dt)^4 \tag{5.10}$$

To complete the proof it remains to compute the coefficient $\beta^{\varphi}_{0,2}(t_0)$. For this we will use the recursive formula (4.30), where B_i are defined by (4.29) with $\beta^{\varphi}_{0,i}$ instead of $\beta_{0,2}$. From (4.30) it follows that

$$B_2(t) = \frac{1}{4}B_1'(t),$$

$$B_3(t) = \frac{1}{5}\left(B_2'(t) - (B_1(t))^2\right) = \frac{1}{20}B_1''(t) - \frac{1}{5}(B_1(t))^2 \tag{5.11}$$

From (4.28), (4.29), and (5.4) it follows that

$$B_1(t) = -\beta^{\varphi}_{0,0}(t) = -\frac{1}{3}\mathbb{S}\left(\varphi(t)\right) = -\frac{\rho(t)}{k}$$

Then by (4.30) and (5.11)

$$\beta^{\varphi}_{0,2}(t) = -3B_3(t) = \frac{3}{20k}\rho''(t) + \frac{3}{5k^2}\rho(t)^2$$

This together with (5.10) implies (5.8). To obtain (5.9) we just rewrite (5.8), taking into account the connection between the function $g(t_0, t_1)$ and the functions $\rho(t)$ $(= \beta_{0,0}(t))$, $\beta_{0,2}(t)$ given by expansion (4.16)(the expression (5.9) is just the most symmetric w.r.t. t_0 and t_1). $\Box$

If $A(t)$ does not change sign, then the canonical length element $|A(t)|^{\frac{1}{4}}dt$ is defined on $\Lambda(\cdot)$. The corresponding parameter τ (i.e., length with respect to this length element) is called *a normal parameter* (in particular, it implies that abnormal extremals may have canonical (normal) parametrization). Calculating the Ricci curvature $\rho_n(\tau)$ of $\Lambda(\cdot)$ in the normal parameter, we obtain a functional invariant of the unparametrized curve. We will call it *projective curvature* of the unparametrized curve $\Lambda(\cdot)$. If $t = \varphi(\tau)$ is the transition function between a projective parameter t and the normal parameter τ, then by (5.4) it follows that $\rho_n(\tau) = \frac{k}{3}\,\mathbb{S}\left(\varphi(\tau)\right)$.

At the end of this section we give an explicit formula for the fundamental form of the regular curve in terms of its curvature operator. First note that by definition the weight k of the regular curve is equal to $\frac{1}{2}\dim W$ (one can also derive it from (3.2) and (4.21)).

Lemma 5.2 *The fundamental form $\mathcal{A}$ of the regular curve $\Lambda(t)$ in the Lagrange Grassmannian $L(W)$ satisfies the following relation*

$$\mathcal{A} = \frac{1}{15}\left(\operatorname{tr}\left(R(t)^2\right) - \frac{1}{k}\left(\operatorname{tr} R(t)\right)^2\right)(dt)^4, \tag{5.12}$$

where $R(t)$ is the curvature operator of $\Lambda(t)$ defined by (3.1) and $k = \frac{1}{2}\dim W$.

Proof. Let us compute $\beta_{0,2}(t)$. We will use the notation of (2.2) and (2.3). By (4.22)

$$\beta_{0,2}(t) = -3\mathrm{tr}\,(Q_3(t)\dot{\Lambda}(t)) = -3\mathrm{tr}\,(A_3(t)\dot{S}(t)) \tag{5.13}$$

For a given $\bar{t}$ choose for simplicity a coordinate representation S_t of the curve $\Lambda(t)$ such that $A_0(\bar{t}) = 0$. Then by (3.3)

$$A_3(\bar{t}) = \frac{1}{5}\left(\dot{A}_2(\bar{t}) - A_1(\bar{t})\dot{S}_{\bar{t}}A_1(\bar{t})\right) \tag{5.14}$$

From (3.4) it follows that the condition $A_0(\bar{t}) = 0$ is equivalent to $\ddot{S}_{\bar{t}} = 0$. It implies that $\dot{A}_2(\bar{t})\dot{S}_{\bar{t}} = \frac{d}{dt}(A_2(t)\dot{S}_t)|_{t=\bar{t}}$. Therefore multiplying (5.14) by $S_{\bar{t}}$ and taking trace from both sides we obtain

$$\mathrm{tr}\,\left(A_3(\bar{t})\dot{S}_{\bar{t}}\right) = \frac{1}{5}\frac{d}{dt}\mathrm{tr}\,\left(A_2(\bar{t})\dot{S}_{\bar{t}}\right) - \frac{1}{5}tr\,\left((A_1(\bar{t})\dot{S}_{\bar{t}})^2\right) \tag{5.15}$$

Now by (4.22) and (4.18)

$$\mathrm{tr}\,\left(A_2(\bar{t})\dot{S}_{\bar{t}}\right) = -\frac{1}{2}\beta_{0,1}(\bar{t}) = -\frac{1}{4}\rho'(\bar{t}) \tag{5.16}$$

On the other hand by (3.6) $(A_1(\bar{t})\dot{S}_{\bar{t}}) = -\frac{1}{3}R(\bar{t})$. This and (5.16) imply that (5.15) can be written in the following form

$$\mathrm{tr}\,\left(A_3(\bar{t})\dot{S}_{\bar{t}}\right) = -\frac{1}{20}\rho''(\bar{t}) - \frac{1}{45}tr\,\left(R(\bar{t})^2\right)$$

Taking into account (5.14) we obtain by (5.8) that

$$A(\bar{t}) = \frac{3}{20}\rho''(\bar{t}) + \frac{1}{15}tr\,\left(R(\bar{t})^2\right) - \frac{3}{5k}\rho(\bar{t})^2 - \frac{3}{20}\rho''(\bar{t}) = \frac{1}{15}tr\,\left(R(\bar{t})^2\right) - \frac{3}{5k}\rho(\bar{t})^2$$

Finally, note that $\rho = \frac{1}{3}\,\mathrm{tr}R$ (see (4.27)). Substituting this to the last relation we obtain (5.12).□

Note that in the scalar case (i.e., when $\dim W = 2$) the fundamental form $\mathcal{A}$ is identically equal to zero.

Remark 2 *All constructions of sections* 3 - 5 *can be done for the curve in the Grassmannian* $G(m, 2m)$ *(the set of all* m*-dimensional subspaces in the* $2m$*-dimensional linear space) instead of Lagrangian Grassmannian by the action of the group* $GL(2m)$ *instead of Symplectic Group.*

6 The rank 1 curves: preliminary steps.

In the present section we start a systematic study of the curves of rank 1 in the Lagrange Grassmannian $L(W)$ with $\dim W = m$. We consider a rank 1 ample curve $\Lambda : I \mapsto L(W)$ with a maybe nonconstant weight, where I is some interval on the real line. We introduce a canonical basis on each subspace $\Lambda(t)$ and compute some characteristics of the curve, in particular, its weight at any point. Finally, we show that the curve Λ has the constant weight equal to m^2 on the set with discrete complement in I. All this will prepare us to the next section, where the curves of rank 1 and constant weight will be investigated.

Without loss of generality, suppose that $\Lambda(\tau)$ is monotonically nondecreasing, i.e., the velocities $\dot\Lambda(t)$ are nonnegtive definite quadratic forms. As in section 2, let $\Lambda_\tau(t)$ be the identical imbedding of $\Lambda(t)$ in the affine space $\Lambda(\tau)^{\pitchfork}$. The velocity $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is well defined self-adjoint linear mapping from $\Lambda(\tau)^*$ to $\Lambda(\tau)$, i.e., an element of $Sym^2\Lambda(\tau)$. Moreover, by our assumptions, $\frac{\partial}{\partial t}\Lambda_\tau(t)$ is a nonpositive self-adjoint linear mapping of rank 1. So for $t \neq \tau$ there exists a unique, up to the sign, vector $w(t,\tau) \in \Lambda(\tau)$ such that for any $v \in \Lambda(\tau)^*$

$$\langle v, \frac{\partial}{\partial t}\Lambda_\tau(t)v\rangle = -\langle v, w(t,\tau)\rangle^2 \tag{6.1}$$

Remark 3 *From the definition of $w(t,\tau)$ it follows easily that for given τ the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely by $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$. Since the Symplectic Group acts transitively on the set of pairs of transversal Lagrange subspaces, one can conclude that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely, up to a symplectic transformation, by the germ of the function $t \mapsto w(t,\tau)$ at $t=\tau$.*

The function $t \mapsto \Lambda_\tau(t)$ has a pole at $t=\tau$. It implies easily that the function $t \mapsto w(t,\tau)$ also has a pole at $t=\tau$. Suppose that the order of this pole is equal to $l(\tau)$.

Denote by $u(t,\tau)$ the normalized curve $t \to u(t,\tau) = (t-\tau)^{l(\tau)}w(t,\tau)$ and define the following vectors in $\Lambda(\tau)$:

$$e_j(\tau) = \frac{1}{(j-1)!}\,\frac{\partial^{j-1}}{\partial t^{j-1}}u(t,\tau)\Big|_{t=\tau}. \tag{6.2}$$

First note that

$$\mathrm{span}\left(\{e_j(\tau)\}_{j=1}^{\infty}\right) = \Lambda(\tau) \tag{6.3}$$

Otherwise, using the following formula

$$w(t,\tau)=\sum_{i=1}^{j} e_i(\tau)(t-\tau)^{i-1-l}+O((t-\tau)^{i-l})$$

one can easily obtain the contradiction to the fact that $\Lambda(t)$ is ample.

Thus for a given parameter τ and integer i, $1\le i\le m$ the following integers $k_i(\tau)$ are well defined

$$k_i(\tau)=\min\{j\in\mathbb{N}\cup 0 : \dim\left(\mathrm{span}\left(e_1(\tau),e_2(\tau),\dots,e_{j+1}(\tau)\right)\right)=i\} \tag{6.4}$$

Note that

$$0=k_1(\tau)<k_2(\tau)<\ldots<k_m(\tau),\;\; k_i(\tau)\ge i-1 \tag{6.5}$$

By definition the vectors $e_{k_1(\tau)+1}(\tau),\dots,e_{k_m(\tau)+1}(\tau)$ constitute the basis of the subspace $\Lambda(\tau)$. We call this basis a *canonical basis* of $\Lambda(\tau)$. Since the vector $w(t,\tau)$ is defined up to the sign, the vector $e_1(\tau)$ $(=e_{k_1(\tau)+1}(\tau))$ is also defined up to the sign. So, one can take also $(-e_{k_1(\tau)+1}(\tau),\dots,-e_{k_m(\tau)+1}(\tau))$ as the canonical bases on the plane $\Lambda(\tau)$. Denote by $w_i(t,\tau)$ the i-th component of the vector $w(t,\tau)$ w.r.t this basis. In other words, functions $w_i(t,\tau)$ satisfies the following relation

$$w(t,\tau)=\sum_{i=1}^{m} w_i(t,\tau)e_{k_i(\tau)+1}(\tau) \tag{6.6}$$

Remark 4 *Using Remark 3, one can easily conclude that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely by $\Lambda(\tau)$, the canonical basis in $\Lambda(\tau)$, the derivative subspace $\Lambda^0(\tau)$, and the germs of the functions $t\mapsto w_i(t,\tau)$ at $t=\tau$, where $1\le i\le m$. Since for any two pairs (Λ,Δ) and $(\tilde\Lambda,\tilde\Delta)$ of transversal Lagrange subspaces with fixed bases in Λ and $\tilde\Lambda$, there exists symplectic transformation that transforms basis in Λ to the basis in $\tilde\Lambda$ and subspace Δ to $\tilde\Delta$, we have that the germ of the curve $\Lambda(t)$ at $t=\tau$ is defined uniquely, up to a symplectic transformation, by the germ of the functions $t\mapsto w_i(t,\tau)$ at $t=\tau$, where $1\le i\le m$.*

Now we prove a computational lemmas about the weight of $\Lambda(t)$ at τ and the order of pole of $t\mapsto w(t,\tau)$.

Lemma 6.1 *The order $l(\tau)$ of pole of the function $t\mapsto w(t,\tau)$ is equal to $k_m(\tau)+1$. The weight of the curve $\Lambda(t)$ at τ is equal to $(2\,k_m(\tau)+1)\,m-2\sum_{i=2}^{m}k_i(\tau)$.*

Proof. For simplicity we will write k_i instead of $k_i(\tau)$, and l instead of $l(\tau)$. Let S_t, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$, be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_{k_1+1}(\tau), \dots, e_{k_m+1}(\tau)$ constitute a standard basis of $\mathbb{R}^m$. Denote by $\Delta = 0 \oplus \mathbb{R}^m$, the subspace $D \in \Lambda(\tau)^{\pitchfork}$. From (6.2) it follows that in the canonical basis

$$w_i(t,\tau) = (t-\tau)^{k_i - l} + O((t-\tau)^{k_i - l + 1}). \tag{6.7}$$

Then relation (6.1) in the canonical basis can be rewritten in the following form

$$\left(\frac{\partial}{\partial t}(S_t - S_\tau)^{-1}\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau) = -(t-\tau)^{k_i+k_j-2l} + O((t-\tau)^{k_i+k_j-2l+1}) \tag{6.8}$$

For simplicity take coordinates $t \mapsto S_t$ such that the subspace Δ is the derivative subspace $\Lambda^0(\tau)$. Then by definition of the derivative subspace the free term on the Laurent expansion of $(S_t - S_\tau)^{-1}$ is equal to zero. Therefore

$$\left((S_t - S_\tau)^{-1}\right)_{i,j} = -\int^t w_i(\xi,\tau)w_j(\xi,\tau)\,d\xi = \frac{(t-\tau)^{2l-k_i-k_j-1}}{k_i+k_j-2l+1} + O((t-\tau)^{k_i+k_j-2l+2}) \tag{6.9}$$

Then it is easy to get the following expansion for the determinant

$$\det(S_t - S_\tau) = \frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1}), \tag{6.10}$$

where

$$k = (2l-1)\,m - 2\sum_{i=2}^{m} k_i, \tag{6.11}$$

and C is the determinant of matrix whose (i,j)th entry is $\frac{1}{2l-k_i-k_j-1}$, $i,j = 1,\dots,m$. It is well known that the determinant of the matrix whose (i,j)th entry is $\frac{1}{x_i+y_j}$, $i,j = 1,\dots,m$, can be computed by the following formula

$$\det\left(\left\{\frac{1}{x_i+y_j}\right\}_{i,j=1}^{m}\right) = \frac{\prod\limits_{1\le i<j\le m}(x_i-x_j)(y_i-y_j)}{\prod\limits_{i,j=1}^{m}(x_i+y_j)} \tag{6.12}$$

It implies in particular that $C \neq 0$ (one can take $x_i = y_i = l - k_i - \frac{1}{2}$ and use the fact that $k_i \neq k_j$ for $i \neq j$). So, the weight is equal to $(2l-1)m - 2\sum_{i=2}^{m} k_i$.

Further, from (6.9) and (6.12) it follows that

$$(S_t - S_\tau)_{i,j} = \left(C_{i,j}(t-\tau)^{-k+2l-k_i-k_j-1} + O((t-\tau)^{-k+2l-k_i-k_j})\right)\left(\frac{(t-\tau)^k}{C} + O((t-\tau)^{k+1})\right) =$$

$$\frac{C_{i,j}}{C}(t-\tau)^{2l-k_i-k_j-1} + O((t-\tau)^{2l-k_i-k_j}), \tag{6.13}$$

where C is as in (6.10), k is as in (6.11), and $C_{i,j}$ are (i,j)th entry of adjacent matrix to the matrix $\left(\frac{1}{2l-k_i-k_j-1}\right)_{i,j=1}^m$. By (6.12) and (6.5) $C_{i,j} \neq 0$. Since S_t is a smooth curve at τ all powers $2l - k_i - k_j - 1$ in (6.13) are positive. By assumption, $\dot{S}_\tau \neq 0$. It implies that

$$\min_{1 \leq i,j \leq m} (2l - k_i - k_j - 1) = 1 \tag{6.14}$$

But from (6.5) it follows that $\min\limits_{1 \leq i,j \leq m} (2l - k_i - k_j - 1) = 2l - 2k_m - 1$ that yields that $l = k_m + 1$. Consequently the weight is equal to $(2\,k_m + 1)\,m - 2\sum_{i=2}^m k_i$. $\square$

Remark 5 *In the proof of the previous lemma to obtain the asymptotics (6.9) we have taken the coordinate representation* $t \mapsto S_t$, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, *with* $\Delta = \Lambda^0(\tau)$ *(where* $\Delta = 0 \oplus \mathbb{R}^m$*). But then we have obtained the relation (6.14)which implies that* $k_i(\tau) + k_j(\tau) - 2l(\tau) + 1 < 0$ *for any* $i, j = 1, \dots, m$. *Therefore, the asymptotics (6.9) for* $\left((S_t - S_\tau)^{-1}\right)_{i,j}$ *and then the asymptotics (6.13) for* $(S_t - S_\tau)_{i,j}$ *are valid for any coordinate representation* $t \mapsto S_t$, $\Lambda(t) = \{(x, S_t\,x) : x \in \mathbb{R}^m\}$, *of germ of* $\Lambda(t)$ *at* $t = \tau$ *such that the canonical basis* $e_{k_1+1}(\tau), \dots, e_{k_m+1}(\tau)$ *constitute a standard basis of* $\mathbb{R}^m$ *and* $\Delta = 0 \oplus \mathbb{R}^m$ *is arbitrary subspace transversal to* $\Lambda(\tau)$. *The reason is that the asymptotics (6.9) do not depend on free term.*

Take some subspace $\Delta \in \Lambda(\tau)^\pitchfork$. Recall that the velocity $\dot{\Lambda}(t)$ is a self-adjoint nonnegative definite linear mapping of rank 1 from $\Lambda(t)$ to $\Lambda(t)^*$. For any t sufficiently closed to τ one can identify Δ with $\Lambda(t)^*$. Under this identification $\dot{\Lambda}(t)$ is a self-adjoint nonnegative linear mapping of rank 1 from $\Lambda(t)$ to Δ. Therefore there exist a unique, up to the sign, vector $v(t) \in \Delta$ such that for any $w \in \Lambda(t)$:

$$\langle \dot{\Lambda}(t)w, w\rangle = \langle v(t), w\rangle^2 \tag{6.15}$$

Suppose that a tuple of vectors $f_1(\tau), \dots, f_m(\tau)$ is a basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (i.e, $\sigma(f_i(\tau), e_{k_j(\tau)+1}(\tau)) = \delta_{i,j}$). From Remark 5 and relation (6.13) (where $l = k_m + 1$) it follows that the components $v_i(t)$ of the vector $v(t)$ w.r.t the basis $f_1(\tau), \dots, f_m(\tau)$ have the following asymptotics

$$v_i(t) = c_i(\tau)(t-\tau)^{k_m(\tau)-k_i(\tau)} + O\Big((t-\tau)^{k_m(\tau)-k_i(\tau)+1}\Big), \tag{6.16}$$

where $c_i(\tau) \neq 0$. (Actually, using (6.12), one can compute $c_i(\tau)$:

$$c_i(\tau) = \sqrt{\frac{C_{i,i}(\tau)(2(k_m(\tau)-k_i(\tau))+1)}{C(\tau)}} = \frac{\prod\limits_{j=1}^{m}(2k_m(\tau)-k_i(\tau)-k_j(\tau)+1)}{\prod\limits_{1\leq j\leq m,\, j\neq i}(k_i(\tau)-k_j(\tau))} \tag{6.17}$$

where $C_{i,i}$ and C are as in the proof of Lemma 6.1.) The relation (6.16) implies that for any integer nonnegative j such that

$$k_m(\tau) - k_i(\tau) \leq j < k_m(\tau) - k_{i-1}(\tau) \tag{6.18}$$

the following relation holds

$$\mathrm{span}(v(\tau), v'(\tau), \dots, v^{(j)}(\tau)) = \mathrm{span}(f_i(\tau), \dots, f_m(\tau)) \tag{6.19}$$

In particular,

$$\begin{aligned} \mathrm{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau)-1)}(\tau)) &= \mathrm{span}(f_2(\tau), \dots, f_m(\tau)) \subsetneq \Delta, & (6.20)\\ \mathrm{span}(v(\tau), v'(\tau), \dots, v^{(k_m(\tau))}(\tau)) &= \Delta & (6.21)\end{aligned}$$

(recall that $k_1(\tau) = 0$).

Now we are ready to prove the following

Proposition 3 *For the ample curve $\Lambda : I \mapsto L(W)$ of rank 1 the set $\mathcal{C}$ such that*

$$\mathcal{C} = \Big\{t \in I : \dim\Big(\mathrm{span}\,(e_1(t), e_2(t), \dots, e_m(t))\Big) < m\Big\} \tag{6.22}$$

is discrete set of the interval of definition I.

Proof. Suppose that $\mathcal{C}$ has an accumulation point τ. Take some subspace $\Delta \in \Lambda(\tau)^{\pitchfork}$. Let $t \mapsto v(t)$ be a curve of vectors in Δ defined by (6.15) for all t from some neighborhood U of τ in I. Note that $t \in \mathcal{C}$ iff $k_m(t) \geq m$. Therefore, by (6.20) and (6.21), we have that $t_0 \in \mathcal{C} \cap U$ iff the function $d(t) \stackrel{def}{=} \det(v(t), v'(t), \dots v^{(m-1)}(t))$ has zero at $t = t_0$. For the accumulation point τ , using consequently the Rolle theorem, one can conclude that the function $d(t)$ has zero of infinite order at $t = \tau$.

On the other hand, let $l_i(\tau) \stackrel{def}{=} k_m(\tau) - k_{m-i+1}(\tau)$. Denote by $p = \sum_{i=1}^m l_i(\tau)$. Let us prove that $d^{(p)}(\tau)$ is not equal to zero. Indeed, $d^{(p)}(\tau)$ can be expressed as the sum of the terms of the form $det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau))$, where

$$\sum_{i=1}^m j_i = p, \quad 0 \le j_1 < j_2 < \dots < j_m. \tag{6.23}$$

Let us show that if the tuple $(j_1, \dots, j_m)$ is different from the tuple $(l_1(\tau), \dots, l_m(\tau))$ and satisfies (6.23), then

$$\det(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau)) = 0. \tag{6.24}$$

For this note first that by assumptions there exists an index s, such that $j_s < l_s(\tau)$ $(= k_m(\tau) - k_{m-s+1}(\tau))$. Then from (6.18) and (6.19) we have the following relation

$$\mathrm{span}(v^{(j_1)}(\tau), \dots, v^{(j_s)}(\tau)) \subset \mathrm{span}(f_{m-s+2}(\tau), \dots, f_m(\tau)),$$

i.e., $\dim\Big(\mathrm{span}(v^{(j_1)}(\tau), \dots, v^{(j_s)}(\tau))\Big) < s$. It implies that $\dim\Big(\mathrm{span}(v^{(j_1)}(\tau), \dots, v^{(j_m)}(\tau))\Big) < m$ that is equivalent to (6.24). Note also that from (6.18) and (6.19) it follows easily that $\mathrm{span}\Big(v^{(l_1(\tau))}(\tau), \dots, v^{(l_m(\tau))}(\tau)\Big) = \Delta$. Therefore,

$$d^{(p)}(\tau) = c\det(v^{(l_1(\tau))}(\tau), \dots, v^{(l_m(\tau))}(\tau)) \neq 0$$

(here c is some natural number). Hence $d(t)$ has zero of finite order at $t = \tau$. We obtain the contradiction. $\square$

For $t \in I \backslash \mathcal{C}$ the numbers $k_i(t) = i - 1$. As a consequence of the previous proposition and the expression for the weight from the Lemma 6.1, we obtain the following

Corollary 1 *The ample curve $\Lambda : I \mapsto L(W)$ of rank 1 has the constant weight equal to m^2 on the set with discrete complement in I.*

At the end of this section we give the explicit formula for the velocity $\dot{\Lambda}(\tau)$ in the canonical basis. Let $(e_1^*(\tau), \dots, e_m^*(\tau))$ be a basis in $\Lambda(\tau)^*$ dual to the canonical basis in $\Lambda(\tau)$. As we have seen at the end of the proof of the Lemma 6.1, the (m, m)th entry is the only nonzero entry of the matrix $\dot{S}_\tau$ and it is equal to $\frac{C_{m,m}(\tau)}{C(\tau)} = c_m^2(\tau)$ (where $c_m(\tau)$ is as in (6.17)). Therefore, we obtain the following

Lemma 6.2 *For any $v_1, v_2 \in \Lambda(\tau)$ the following relation holds*

$$\langle \dot{\Lambda}(\tau)v_1, v_2\rangle = c_m^2(\tau)\langle e_m^*(\tau), v_1\rangle\langle e_m^*(\tau), v_2\rangle, \tag{6.25}$$

where

$$c_m(\tau) = \prod_{j=1}^{m-1} \frac{k_m(\tau) - k_j(\tau) + 1}{k_m(\tau) - k_j(\tau)} \tag{6.26}$$

7 The rank 1 curves with a constant weight.

The present section is devoted to the curves of rank 1 and a constant finite weight in the Lagrange Grassmannian $L(W)$. We show that in this case the function $g(t,\tau)$ constructed in section 4 determines the curve uniquely, up to symplectic transformation. We also find a complete system of invariants of the curve in terms of the function g.

First, using Proposition 3 and Lemmas 6.1, we obtain the following

Proposition 4 *If $\Lambda(t)$ is a curve of rank 1 and constant weight on I, then for all $t \in I$ and $1 \le i \le m$ the numbers $k_i(t)$ are equal to $i-1$, or, equivalently, the vectors $e_1(t), \dots, e_m(t)$ constitute the canonical basis of the subspace $\Lambda(t)$.*

Proof. From (6.5) it follows that always

$$k_i(t) - k_j(t) \ge i - j,\ k_1(t) = 0 \tag{7.1}$$

Therefore by Lemma 6.1 the weight $k(t)$ of the curve Λ at the point t satisfies

$$k(t) = (2k_m(t)+1)m - 2\sum_{i=2}^{m} k_i(t) = 2\sum_{i=1}^{m}(k_m(t) - k_i(t)) + m \ge 2\sum_{i=1}^{m}(m-i) + m = m^2 \tag{7.2}$$

In addition, from (7.1) it is easy to see that the equality in (7.2) holds iff $k_i(t) = i-1$ for any $1 \le i \le m$. Therefore, if the set $\mathcal{C}$ is as in (6.22), then for any $t \in \mathcal{C}$ the weight $k(t) > m^2$, while for $t \notin \mathcal{C}$ the weight $k(t) = m^2$. But from Proposition 3 the set $\mathcal{C}$ is discrete subset of I. Hence, for the weight $k(t)$ to be constant on I, the set $\mathcal{C}$ has to be empty. This completes the proof of the Proposition. $\square$

As a consequence of the previous proposition and Lemmas 6.1, 6.2 , we obtain easily the following

Corollary 2 *If $\Lambda(t)$ is a curve of rank 1 and constant weight on I, then:*

1) *at any point $t \in I$ the weight is equal to m^2;*

2) *for any $\tau \in I$ the function $t \mapsto w(t,\tau)$ has a pole of order m at $t = \tau$;*

3) *For any $v_1, v_2 \in \Lambda(\tau)$ the following relation holds*

$$\langle \dot{\Lambda}(\tau) v_1, v_2 \rangle = m^2 \langle e_m^*(\tau), v_1 \rangle \langle e_m^*(\tau), v_2 \rangle \tag{7.3}$$

Now we prove that the function $g(t,\tau)$ defined in section 4 contains all the information about $\Lambda(t)$.

Theorem 1 *The function $g(t,\tau)$ determines the curve $\Lambda(t)$ of rank 1 and constant weight uniquely , up to a symplectic transformation.*

Before starting to prove the theorem, we want to describe in few words the main steps of the proof. First, we show that the function $g(t,\tau)$ is almost the same as the component $w_m(t,\tau)$ of the vector $w(t,\tau)$. The vector $w(t,\tau)$ is a function of two variables, but it is determined by a curve. Therefore it is natural to expect that $w(t,\tau)$ satisfies some partial differential equation. We find this equation that is actually the system of m equations for the components $w_i(t,\tau)$, $1 \leq i \leq m$. Then we show that this system has a "triangular" form such that all components $w_i(t,\tau)$ can be expressed by $w_m(t,\tau)$ and refer to Remark 4 to complete the proof.

Proof of Theorem 1.

1. We start the proof with the following lemma

Lemma 7.1 *The following relation holds*

$$w_m^2(t,\tau) = \frac{1}{(t-\tau)^2} + \frac{1}{m^2} g(t,\tau) \tag{7.4}$$

Proof. By (4.19),(4.23), and part 2) of Corollary 4 we have

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\Lambda_\tau(t) \circ \dot{\Lambda}(\tau)\right) = -\frac{m^2}{(t-\tau)^2} - g(t,\tau) \tag{7.5}$$

Let $t \to S_t$, $\Lambda(t) = \{(x, S_t x) : x \in \mathbb{R}^m\}$ be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau$ such that the canonical basis $e_1(\tau), \dots, e_m(\tau)$ constitute a standard basis of $\mathbb{R}^m$. By (4.24)

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\ \dot{S}_\tau\right) = -\frac{m^2}{(t-\tau)^2} - g(t,\tau) \tag{7.6}$$

Relation (7.3) implies that in the chosen coordinates

$$\dot{S}_\tau = \begin{cases} 0 & (i,j) \neq (m,m) \\ m^2 & (i,j) = m \end{cases}$$

By construction,

$$\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\right)_{i,j} = -w_i(t,\tau)w_j(t,\tau)$$

Therefore

$$\operatorname{tr}\left(\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right)\dot{S}_\tau\right) = -m^2 w_m^2(t,\tau).$$

This together with (7.6) implies (7.4). $\square$

By (7.4) it follows that in order to prove the theorem it is sufficient to show that the function $w_m(t,\tau)$ determines $\Lambda(t)$ uniquely, up to a symplectic transformation.

2. Now we derive a partial differential equation for the vector function $w(t,\tau)$.

Lemma 7.2 *The vector function $w(t,\tau)$ satisfies the following differential equation*

$$\frac{\partial^2 w}{\partial t \partial \tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w}{\partial \tau} + m^2 w_m^2 w = 0 \tag{7.7}$$

Proof. Fix some parameter τ_0 and take some subspace Δ transversal to $\Lambda(\tau_0)$. Let $t \to S_t$, $\Lambda(t) = \{(x, S_t\, x) : x \in \mathbb{R}^m\}$ be a coordinate representation of germ of $\Lambda(t)$ at $t = \tau_0$ such that $\Lambda(\tau_0) = \mathbb{R}^m \oplus 0$ and $\Delta = 0 \oplus \mathbb{R}^m$. Denote by $w^\Delta(t,\tau) \in \mathbb{R}^m$ the first m components of the vector $w(t,\tau)$ in the chosen coordinates (or equivalently, the image of $w(t,\tau)$ under the projection of W on $\Lambda(\tau_0)$ parallel to Δ). Also, let, as before, $f_1(\tau), \dots, f_m(\tau)$ be the basis of Δ dual to the canonical basis of $\Lambda(\tau)$ (w.r.t. the symplectic form σ).

By (6.1) it follows that for t and τ closed to τ_0

$$\frac{\partial}{\partial t}\left((S_t - S_\tau)^{-1}\right) = -w^\Delta(t,\tau)w^\Delta(t,\tau)^T. \tag{7.8}$$

Therefore

$$\dot{S}_t = \Big((S_t - S_\tau)w^\Delta(t,\tau)\Big)\Big((S_t - S_\tau)w^\Delta(t,\tau)\Big)^T \tag{7.9}$$

It implies that the vector function $(S_t - S_\tau)w^\Delta(t,\tau)$ does not depend on τ. Differentiating it w.r.t. τ we get

$$-\dot{S}_\tau w^\Delta(t,\tau) + (S_t - S_\tau)\frac{\partial}{\partial\tau}w^\Delta(t,\tau) = 0 \tag{7.10}$$

From (6.25) it follows that

$$\dot{S}_\tau w^\Delta(t,\tau) = m^2 w_m(t,\tau) f_m(\tau) \tag{7.11}$$

This together with (7.10) implies that

$$\frac{\partial}{\partial\tau}w^\Delta(t,\tau) = m^2 w_m(t,\tau)(S_t - S_\tau)^{-1} f_m(\tau) \tag{7.12}$$

In particular,

$$m^2 (S_t - S_\tau)^{-1} f_m(\tau) = \frac{1}{w_m(t,\tau)}\frac{\partial}{\partial\tau}w^\Delta(t,\tau) \tag{7.13}$$

Now, differentiating (7.12) w.r.t. t, we have

$$\frac{\partial^2}{\partial t\partial\tau}w^\Delta(t,\tau) = m^2 w_m(t,\tau)\frac{\partial}{\partial t}\Big((S_t - S_\tau)^{-1}\Big) f_m(\tau) + \frac{\partial}{\partial t}w_m(t,\tau) m^2 (S_t - S_\tau)^{-1} f_m(\tau) \tag{7.14}$$

From (7.8) it follows that

$$(S_t - S_\tau)^{-1} f_m(\tau) = -w_m(t,\tau) w^\Delta(t,\tau)$$

Substituting this and (7.13) in (7.14), we get

$$\frac{\partial^2 w^\Delta}{\partial t\partial\tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w^\Delta}{\partial\tau} + m^2 w_m^2 w^\Delta = 0 \tag{7.15}$$

Recalling the definition of $w^\Delta(t,\tau)$, we obtain from the last equation the following inclusion

$$\frac{\partial^2 w}{\partial t\partial\tau} - \left(\frac{\frac{\partial w_m}{\partial t}}{w_m}\right)\frac{\partial w}{\partial\tau} + m^2 w_m^2 w \in \Delta \tag{7.16}$$

Let us remember that all our considerations (and in particular the inclusion (7.16)) are valid for any subspace Δ transversal to $\Lambda(\tau_0)$ and any t, τ closed to τ_0. Taking as Δ in (7.16) two subspaces that are transversal to $\Lambda(\tau_0)$ and also transversal one to another, we obtain (7.7) for any t, τ closed to τ_0 . Since τ_0 is arbitrary, this completes the proof of the lemma. $\square$

In the sequel it will be convenient also to make the following substitution in (7.7)

$$Y(t,\tau)=\frac{1}{w_m(t,\tau)}w(t,\tau) \tag{7.17}$$

Then by direct computation one can obtain the following equation for Y:

$$\frac{\partial^2 Y}{\partial t\partial\tau}+\left(\frac{\frac{\partial w_m}{\partial\tau}}{w_m}\right)\frac{\partial Y}{\partial t}+\left(\frac{\partial^2}{\partial t\partial\tau}(\ln w_m)+m^2w_m^2\right)Y=0 \tag{7.18}$$

3. Now we shall rewrite the equation (7.7) as a system of equations w.r.t. the components $w_i(t,\tau)$. Take some subspace $\Delta\in\Lambda(\tau)^{\pitchfork}$. Identifying Δ with $\Lambda(\tau)^*$, denote $f_i(\tau)$ the vector, corresponding to $e_i^*(\tau)$ under this identification. The vectors $e_1(\tau),\dots,e_m(\tau),f_1(\tau),\dots,f_m(\tau)$ constitute the basis of the symplectic space W. Suppose that

$$\dot e_i(\tau)=\sum_{j=1}^m\alpha_{i,j}(\tau)e_j(\tau)+\gamma_{i,j}(\tau)f_j(\tau)$$

According to (7.3)

$$\gamma_{ij}(\tau)=\begin{cases}0 & (i,j)\neq(m,m)\\ m^2 & (i,j)=(m,m)\end{cases}$$

It implies that

$$\begin{aligned}\dot e_i(\tau)&=\sum_{j=1}^m\alpha_{i,j}(\tau)e_j(\tau),\ 1\le i\le m-1,\\ \dot e_m(\tau)&=\sum_{j=1}^m\alpha_{m,j}(\tau)e_j(\tau)+m^2f_m(\tau)\end{aligned} \tag{7.19}$$

Remark 6 *In particular, it follows that the functions $\alpha_{i,j}(\tau)$ with $1\le i\le m-1$ do not depend on the choice of the subspace Δ.*

By definition

$$w(t,\tau)=\sum_{i=1}^m w_i(t,\tau)e_i(\tau).$$

Then, using (7.19), we obtain

$$\frac{\partial w}{\partial\tau}=\sum_{i=1}^m\left(\frac{\partial w_i}{\partial\tau}+\sum_{j=1}^m w_j\alpha_{j,i}\right)e_i+m^2w_mf_m, \tag{7.20}$$

$$\frac{\partial^2 w}{\partial t\partial\tau}=\sum_{i=1}^m\left(\frac{\partial^2 w_i}{\partial t\partial\tau}+\sum_{j=1}^m\frac{\partial w_j}{\partial t}\alpha_{j,i}\right)e_i+m^2\frac{\partial w_m}{\partial t}f_m \tag{7.21}$$

Substituting (7.20) and (7.21) in (7.7) and comparing coefficients of e_i for $i = 1, \dots, m$, we get the following system of equations

$$\frac{\partial^2 w_i}{\partial t \partial \tau} - \frac{\frac{\partial w_m}{\partial t}}{w_m}\frac{\partial w_i}{\partial \tau} + m^2 w_m^2 w_i = \sum_{j=1}^{m}\left(\frac{\frac{\partial w_m}{\partial t}}{w_m} w_j - \frac{\partial w_j}{\partial t}\right)\alpha_{j,i},\ 1 \le i \le m \tag{7.22}$$

The term in the righthand side of (7.22), corresponding to $j = m$, is equal to zero. Hence the equation (7.22) can be written in the following form

$$\frac{\partial^2 w_i}{\partial t \partial \tau} - \frac{\frac{\partial w_m}{\partial t}}{w_m}\frac{\partial w_i}{\partial \tau} + m^2 w_m^2 w_i = \sum_{j=1}^{m-1}\left(\frac{\frac{\partial w_m}{\partial t}}{w_m} w_j - \frac{\partial w_j}{\partial t}\right)\alpha_{j,i},\ 1 \le i \le m \tag{7.23}$$

By Remark 6 the system of equations (7.23) does not depend on the choice of the subspace Δ.

In the same way the equation (7.17) can be rewritten as an equation for components $Y_i(t,\tau) = \frac{w_i(t,\tau)}{w_m(t,\tau)}$ of the vector $Y_i(t,\tau)$ w.r.t the canonical basis:

$$\frac{\partial^2 Y_i}{\partial t \partial \tau} + \left(\frac{\frac{\partial w_m}{\partial \tau}}{w_m}\right)\frac{\partial Y_i}{\partial t} + \left(\frac{\partial^2}{\partial t \partial \tau}(\ln w_m) + m^2 w_m^2\right) Y_i = -\sum_{j=1}^{m-1}\frac{\partial Y_j}{\partial t}\alpha_{j,i} \tag{7.24}$$

4. Now we show that equation (7.23) (or (7.24)) has a "triangle" form. Note that by construction all functions $t \mapsto w(t,\tau)$ have singularities at $t = \tau$. Moreover, from the part 1 of Corollary 4 it follows that their Laurent expansions at $t = \tau$ have the following form

$$w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+1}} + \varphi_i(t,\tau), \tag{7.25}$$

where $\varphi_i(t,\tau)$ are smooth functions. Using this fact one can obtain the following

Lemma 7.3 *The coefficients $\alpha_{j,i}(\tau)$, $1 \le j \le m-1$, satisfy the following relations*

1. *$\alpha_{j,i}(\tau) \equiv 0$, if $j < i-1$;*
2. *$\alpha_{i-1,i}(\tau) \equiv \frac{(i-1)(2m-i+1)}{m-i+1}$;*
3. *If $i \le j \le m-1$, then $\alpha_{j,i}(\tau)$ can be expressed by $\frac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ with $0 \le k \le i-j$, where $\varphi_m(t,\tau)$ is defined by (7.25).*

Proof. We shall analyze the Laurent expansions of both sides of the equation (7.23). We start with the righthand side. Denote by

$$\Phi_m(t,\tau) = \frac{\partial}{\partial t}\ln\Big(1 + (t-\tau)\varphi_m(t,\tau)\Big) \tag{7.26}$$

Using (7.25), one can obtain the following series of relations

$$\frac{\partial}{\partial t} w_j(t,\tau) = -\frac{m-j+1}{(t-\tau)^{m-j+2}} + O(1); \tag{7.27}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m} = \frac{\partial}{\partial t}\ln w_m(t,\tau) = \frac{\partial}{\partial t}\ln\left(\frac{1}{t-\tau} + \varphi_m(t,\tau)\right) = -\frac{1}{t-\tau} + \Phi_m(t,\tau) \tag{7.28}$$

$$\frac{\frac{\partial}{\partial t} w_m}{w_m} w_j = -\frac{1}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}} + O\left(\frac{1}{t-\tau}\right) \tag{7.29}$$

Therefore the righthand side of (7.23) can be written in the following form

$$\sum_{j=1}^{m-1}\left(\frac{m-j}{(t-\tau)^{m-j+2}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-j+1}}\right)\alpha_{j,i}(\tau) + O\left(\frac{1}{t-\tau}\right) \tag{7.30}$$

Suppose that the function $t \mapsto \Phi_m(t,\tau)$ has the following expansion into the formal Taylor series at $t=\tau$

$$\Phi_m(t,\tau) \approx \sum_{k=0}^{\infty} c_k(\tau)(t-\tau)^k \tag{7.31}$$

Then by direct computation we have that the righthand side of (7.23) has the following form

$$\frac{\alpha_{1,i}(\tau)}{(t-\tau)^{m+1}} + \sum_{j=2}^{m-1}\frac{(m-j)\alpha_{j,i}(\tau) + \sum\limits_{k=1}^{j-1} c_{j-k-1}(\tau)\alpha_{k,i}(\tau)}{(t-\tau)^{m-j+2}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.32}$$

Now consider the lefthand side of (7.23). Using (7.26) and (7.25) , we obtain the following series of relations

$$\frac{\partial}{\partial \tau} w_i(t,\tau) = \frac{m-i+1}{(t-\tau)^{m-i+2}} + \frac{\partial}{\partial \tau}\varphi_i(t,\tau); \tag{7.33}$$

$$\frac{\partial^2}{\partial t\partial\tau}w_i(t,\tau) = -\frac{(m-i+1)(m-i+2)}{(t-\tau)^{m-i+3}} + O(1); \tag{7.34}$$

$$\frac{\frac{\partial}{\partial t}w_m}{w_m}\frac{\partial}{\partial\tau}w_i(t,\tau) = -\frac{m-i+1}{(t-\tau)^{m-i+3}} + \frac{\Phi_m(t,\tau)}{(t-\tau)^{m-i+2}} - +O\left(\frac{1}{t-\tau}\right) \tag{7.35}$$

$$w_m(t,\tau)w_i(t,\tau) = \frac{1}{(t-\tau)^{m-i+3}} + \frac{2\varphi_m(t,\tau)}{(t-\tau)^{m-i+2}} + \frac{\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.36}$$

Therefore the lefthand side of (7.23) can be written in the following form

$$\frac{m^2-(m-i+1)^2}{(t-\tau)^{m-i+3}} + \frac{1}{(t-\tau)^{m-i+2}}\Big(\Phi_m(t,\tau)+2\varphi_m(t,\tau)\Big) + \frac{\varphi_m^2(t,\tau)}{(t-\tau)^{m-i+1}} + O\left(\frac{1}{(t-\tau)^2}\right) \tag{7.37}$$

Comparing coefficients of (7.32) and (7.37) we have

1. If $m-j+2 > m-i+3$, i.e., $j < i-1$, then $\alpha_{j,i}(\tau) \equiv 0$. This completes the proof of the first part of the lemma;

2. If $m-j+2 = m-i+3$, i.e., $j = i-1$, then $\alpha_{i-1,i}(\tau)(m-i+1) = m^2-(m-i+1)^2$. This completes the proof of the second part of the lemma;

3. If $2 < m-j+2 < m-i+3$, i.e., $i-1 < j < m$, then, taking into account that $\alpha_{k,i}(\tau) \equiv 0$ for $k < i-1$, we obtain

$$(m-j)\alpha_{j,i}(\tau) + \sum_{k=i-1}^{j-1} c_{j-k-1}(\tau)\alpha_{ki}(\tau) = c_{j-i}(\tau) + \frac{2\frac{\partial^{j-i}}{\partial t^{j-i}}\varphi_m(t,\tau)\Big|_{t=\tau}}{(j-i)!} + \frac{\frac{\partial^{j-i+1}}{\partial t^{j-i+1}}\varphi_m^2(t,\tau)\Big|_{t=\tau}}{(j-i+1)!} \tag{7.38}$$

By (7.26) coefficient $c_n(\tau)$ can be expressed by $\frac{\partial^k}{\partial t^k}\varphi_m(t,\tau)\Big|_{t=\tau}$ with $0 \le k \le n$. This together with (7.38) completes the proof of the third part of the lemma. □

By the previous lemma the equation (7.24) can be written in the following form for $2 \le i \le m$

$$\frac{\partial Y_{i-1}}{\partial t} = -\frac{1}{\alpha_{i-1,i}}\left(\frac{\partial^2 Y_i}{\partial t\partial\tau} + \left(\frac{\frac{\partial w_m}{\partial\tau}}{w_m}\right)\frac{\partial Y_i}{\partial t} + \left(\frac{\partial^2}{\partial t\partial\tau}(\ln w_m) + m^2w_m^2\right)Y_i + \sum_{j=i}^{m-1}\frac{\partial Y_j}{\partial t}\alpha_{j,i}\right) \tag{7.39}$$

where $\alpha_{i-1,i} = \frac{(i-1)(2m-i+1)}{m-i+1}$. All terms in the righthand side of (7.39) depends on the functions $Y_j(t,\tau)$ with $j \geq i$. Note also that by (7.25)

$$Y_i(t,\tau) = \frac{w_i(t,\tau)}{w_m(t,\tau)} = \frac{1}{(t-\tau)^{m-i}} \frac{1+(t-\tau)^{m-i+1}\varphi_i(t,\tau)}{1+(t-\tau)\varphi_m(t,\tau)} \tag{7.40}$$

It implies that in the Laurent expansion at $t=\tau$ of the function $t \mapsto Y(t,\tau)$ all coefficients that correspond to nonpositive powers (and in particular the free term) depend on $w_m(t,\tau)$. This together with (7.39) yields that all $Y_i(t,\tau)$ (and therefore all $w_i(t,\tau)$) with $1 \leq i \leq m-1$ can be expressed by $w_m(t,\tau)$. But by Remark 4 the components $w_i(t,\tau)$, $1 \leq i \leq m$ determine the curve $\Lambda(t)$ uniquely, up to a symplectic transformation. This completes the proof of the Theorem 1. $\square$

Now our goal is to find a complete system of symplectic invariants of curve $\Lambda(t)$ of rank 1 and the constant weight, i.e., some set of functions of t which determines $\Lambda(t)$ uniquely, up to a symplectic transformation. By Theorem 1 it is natural to look for a complete system of invariants among coefficients $\beta_{i,j}(t)$ of the expansion (4.16) of g in the Taylor series. Since $\Lambda(t)$ can be described, up to symplectic transformation, by the curve $t \to w(t,\tau)$ of the vectors on the linear space of dimension m, it is natural to expect that complete system of invariants of $\Lambda(\cdot)$ consists of m functions of t. By Lemma 4.1 the first m "independent" coefficients in expansion (4.16) are $\beta_{0,2i}(t)$ with $0 \leq i \leq m-1$. All this arguments lead to the following theorem:

Theorem 2 *The coefficients $\beta_{0,2i}(t)$, $0 \leq i \leq m-1$, determine the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Let a function $\varphi_m(t,\tau)$ be as in (7.25). From the identity (7.4) it follows easily that the theorem is equivalent to the following theorem:

Theorem 2′. *The functions $\tau \mapsto \frac{\partial^{2i-1}\varphi_m(t,\tau)}{\partial t^{2i-1}}\Big|_{t=\tau}$, $1 \leq i \leq m$, determine the curve $\Lambda(t)$ of rank 1 and a constant weight uniquely, up to a symplectic transformation.*

Proof of Theorem 2′. Let functions $\varphi_i(t,\tau)$, $1 \leq i \leq m$, be as in (7.25). First, using the system of equation (7.23), we prove the following lemma:

Lemma 7.4 *Any partial derivatives of the functions $\varphi_i(t,\tau)$,$1 \leq i \leq m$, at any diagonal points (τ,τ) can be expressed by the functions $\tau \mapsto \frac{\partial^{2j-1}\varphi_m}{\partial t^{2j-1}}(t,\tau)\big|_{t=\tau}$ and their derivatives , where $1 \leq j \leq m$.*

Proof. First, it is natural to make the change of coordinates $x = t-\tau$, $y = t+\tau$ such that the diagonal $t=\tau$ becomes the axis $x=0$ in the new coordinates. Indeed, if we denote by $z_i(x,y) = w_i(\frac{x+y}{2}, \frac{y-x}{2})$, then the system (7.23) can be transformed into the following system w.r.t. z_i:

$$-z_m \left(\frac{\partial^2 z_i}{\partial x^2} - \frac{\partial^2 z_i}{\partial y^2} \right) - \left(\frac{\partial z_m}{\partial x} + \frac{\partial z_m}{\partial y} \right) \left(\frac{\partial z_i}{\partial y} - \frac{\partial z_i}{\partial x} \right) + m^2 z_m^3 z_i =$$

$$\sum_{j=i-1}^{m-1} \left(\left(\frac{\partial z_m}{\partial x} + \frac{\partial z_m}{\partial y} \right) z_j - \left(\frac{\partial z_j}{\partial x} + \frac{\partial z_j}{\partial y} \right) z_m \right) \alpha_{ji}, \ 1 \leq i \leq m \tag{7.41}$$

(here we also have used the first part of the Lemma 7.3). Relations (7.25) can be transformed into the following

$$z_i(x,y) = \frac{1}{x^{m-i+1}} u_i(x,y), \tag{7.42}$$

where the functions $u_i(x,y)$ are smooth, $u_i(0,y) \equiv 1$, and $\frac{\partial^k}{\partial x^k} u_i(0,y) = 0$ for $1 \leq i \leq m$, $1 \leq k \leq m-i$. Substitute (7.42) in (7.41) and multiply both sides on x^{m-i+4}. Then we obtain some singular system of equations w.r.t. u_i. By direct calculation it can be shown that this system has the following form:

$$\begin{aligned} &x^2 u_m \tfrac{\partial^2 u_i}{\partial x^2} - (2m-2i+1) x u_m \tfrac{\partial u_i}{\partial x} + (m-i+1) x u_i \tfrac{\partial u_m}{\partial x} + \alpha_{i-1,i} x u_{i-1} \tfrac{\partial u_m}{\partial x} - \\ &\alpha_{i-1,i} x u_m \tfrac{\partial u_{i-1}}{\partial x} + (m-i+1)^2 u_m u_i + \alpha_{i-1,i}(m-i+1) u_{i-1} u_m - m^2 u_m^3 u_i = \Psi_i, \end{aligned} \tag{7.43}$$

where

$$\begin{aligned} \Psi_i &= x^2 u_m \frac{\partial^2 u_i}{\partial y^2} + x u_m \frac{\partial u_i}{\partial y} - (m-i+1) x u_i \frac{\partial u_m}{\partial y} - \alpha_{i-1,i} x u_{i-1} \frac{\partial u_m}{\partial y} + \\ &\alpha_{i-1,i} x u_m \frac{\partial u_{i-1}}{\partial y} - x^2 \left(\frac{\partial u_m}{\partial x} + \frac{\partial u_m}{\partial y} \right) \left(\frac{\partial u_i}{\partial y} - \frac{\partial u_i}{\partial x} \right) - \sum_{j=i}^{m-1} \left(x^{j-i+2} \left(\frac{\partial u_m}{\partial x} + \frac{\partial u_m}{\partial y} \right) u_j - \right. \\ &\left. x^{j-i+2} \left(\frac{\partial u_j}{\partial x} + \frac{\partial u_j}{\partial y} \right) u_m + (m-j) x^{j-i+1} u_m u_j \right) \alpha_{ji} \end{aligned} \tag{7.44}$$

The lefthand side of equation (7.43) is a principal part of this equation in the following sense: Differentiate both sides of (7.43) k times in x at the points of the initial curve $x = 0$. Then the righthand side can be expressed by the partial derivatives $\frac{\partial^n u_p}{\partial x^n}(0,y)$ with n **less than** k and their derivatives w.r.t. y (here one can take $i-1 \leq p \leq m$), while any term of the lefthand side (at least for $k \geq 2$) depends also on partial derivative of some u_j w.r.t. x of **order** k at $(0,y)$. Moreover, using that $u_i(0,y) \equiv 1$, $1 \leq i \leq m$, and $\alpha_{i-1,i} = \frac{(2m-i+1)(i-1)}{m-i+1}$, one can easily obtain in this way the following linear system w.r.t. $\frac{\partial^k}{\partial x^k} u_i(0,y)$, $1 \leq i \leq m$ for a given integer $k \geq 0$:

$$\zeta_i(k) \frac{\partial^k u_{i-1}}{\partial x^k}(0,y) + \eta_i(k) \frac{\partial^k u_i}{\partial x^k}(0,y) + \theta_i(k) \frac{\partial^k u_m}{\partial x^k}(0,y) = \widetilde{\Psi}_i, \ \ 1 \leq i \leq m, \tag{7.45}$$

where

$$
\begin{aligned}
\zeta_i(k) &= \frac{(k+i-m-1)(i-2m-1)(i-1)}{m-i+1} \\
\eta_i(k) &= (k+i-1)(k+i-2m-1) \\
\theta_i(k) &= \frac{k+2i-2-2m}{m-i+1}m^2,
\end{aligned}
\tag{7.46}
$$

and $\widetilde{\Psi}_i$ can be expressed by the partial derivatives of the form $\frac{\partial^n u_p}{\partial x^n}(0,y)$ with n **less than** k and their derivatives w.r.t. y (here $i-1 \le p \le m$).

It turns out that the determinant of the system (7.45) satisfies the following remarkable identity:

$$
\begin{vmatrix}
\eta_1(k) & 0 & 0 & \dots & 0 & 0 & \theta_1(k) \\
\zeta_2(k) & \eta_2(k) & 0 & \dots & 0 & 0 & \theta_2(k) \\
0 & \zeta_3(k) & \eta_3(k) & \dots & 0 & 0 & \theta_3(k) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \dots & \zeta_{m-1}(k) & \eta_{m-1}(k) & \theta_{m-1}(k) \\
0 & 0 & 0 & \dots & 0 & \zeta_m(k) & \eta_m(k)+\theta_m(k)
\end{vmatrix}
= \prod_{j=1}^{m}(k-2j)(k+2j-1)
\tag{7.47}
$$

The proof of (7.47) that we have found is rather long and will be presented in Appendix.

As a consequence of (7.47) we obtain that the determinant of the system (7.45) has exactly m positive zeros at $k=2j$, $1 \le j \le m$. Therefore, any partial derivative of u_i, $1 \le i \le m$, at $(0,y)$ can be expressed by the functions $y \mapsto \frac{\partial^{2j} u_p}{\partial x^{2j}}(0,y)$ and their derivatives, where $1 \le j, p \le m$. Moreover, by Theorem 1 $u_p(x,y)$ can be expressed by $u_m(x,y)$ and its derivative. Hence any partial derivative of u_i, $1 \le i \le m$, at $(0,y)$ can be expressed by the functions $y \mapsto \frac{\partial^{2j} u_m}{\partial x^{2j}}(0,y)$ and their derivatives, where $1 \le j \le m$. But this is equivalent to the statement of our Lemma, if we return to the old coordinates t and τ. □

Now we define a canonical moving frame: For given τ take the derivative subspace $\Lambda^0(\tau)$ and let $f_1(\tau), \dots, f_m(\tau)$ be a basis of $\Lambda^0(\tau)$ dual to the canonical basis of $\Lambda(\tau)$ (i.e., $\sigma(f_i(\tau), e_j(\tau)) = \delta_{i,j}$). The basis $(e_1(\tau), \dots, e_m(\tau), f_1(\tau), \dots, f_m(\tau))$ of whole symplectic space W is called *the canonical moving frame* of the curve $\Lambda(\cdot)$. Denote by $E(\tau)$ and $F(\tau)$ the tuples of vectors $(e_1(\tau), \dots, e_m(\tau))$ and $(f_1(\tau), \dots, f_m(\tau))$ correspondingly , arranged in the columns. Denote by S_t the matrix, corresponding to the linear mapping $\langle \Lambda(\tau), \Lambda(t), \Lambda^0(\tau)\rangle$ w.r.t to the canonical basis, and by S_t^0 the matrix, corresponding to the linear mapping $\langle \Lambda^0(\tau), \Lambda^0(t), \Lambda(\tau)\rangle$ w.r.t to the basis $(f_1(\tau), \dots, f_m(\tau))$

(see section 2 for notations). Also, let $\Omega(\tau)$ be a $m\times m$ matrix with (i,j) entry equal to $\alpha_{i,j}(\tau)$, where $\alpha_{i,j}(\tau)$ is defined by (7.19) with $\Delta=\Lambda^0(\tau)$. Then it is easy to see that the structural equation for the canonical moving frame has the following form:

$$\begin{pmatrix}\dot E(\tau)\\ \dot F(\tau)\end{pmatrix}=\begin{pmatrix}\Omega(\tau) & \dot S_\tau\\ \dot S^0_\tau & -\Omega^T(\tau)\end{pmatrix}\begin{pmatrix}E(\tau)\\ F(\tau)\end{pmatrix}. \tag{7.48}$$

We claim that in order to prove Theorem $2'$ it is sufficient to prove the following lemma:

Lemma 7.5 *The matrix in the structural equation (7.48) depends only on the coefficients of the expansions of $t\to w_i(t,\tau)$, $1\le i\le m$, in the Laurent series at $t=\tau$.*

Indeed, if Lemma 7.5 holds then first by Lemma 7.4 this matrix depends only on the functions $\tau\mapsto\frac{\partial^{2j-1}\varphi_m}{\partial t^{2j-1}}(t,\tau)\big|_{t=\tau}$, $1\le j\le m$, secondly, the structural equation (7.48) has a unique solution with prescribed initial condition, and, finally, any symplectic basis can be taken as an initial condition of (7.48).

Proof of Lemma 7.5. First, according to (7.3)

$$(\dot S_\tau))_{i,j}=\begin{cases}0 & (i,j)\neq(m,m)\\ m^2 & (i,j)=(m,m),\end{cases} \tag{7.49}$$

Further, by recursive formula (2.4) for $i=0$:

$$\dot S^0_\tau=\frac{d}{d\tau}A_0(\tau)=A_1(\tau)+\sum_{n=1-2m}^{-1}\Big(A_n(\tau)\dot S_\tau A_{-n}(\tau)+A_{-n}(\tau)\dot S_\tau A_n(\tau)\Big), \tag{7.50}$$

where $A_j(\tau)$ are defined by expansion (2.3) (here we have used that by definition of the derivative curve $A_0(\tau)=0$ and by Lemma 6.1 the order of pole of $t\mapsto(S_t-S_\tau)^{-1}$ at $t=\tau$ is equal to $2m-1$). By definition of the vectors $w(t,\tau)$, we have

$$\Big((S_t-S_\tau)^{-1}\Big)_{i,j}=-\int^t w_i(\xi,\tau)w_j(\xi,\tau)d\xi.$$

Therefore any $A_n(\tau)$ with $n\neq 0$ can be expressed by the coefficients of the expansions of $t\to w_i(t,\tau)$, $1\le i\le m$, in the Laurent series at $t=\tau$. This together with (7.49) and (7.50) implies that $\dot S^0_\tau$ can be expressed by the coefficients of the expansions of $t\to w_i(t,\tau)$, $1\le i\le m$, in the Laurent series at $t=\tau$.

Finally, let us analyze the matrix $\Omega(\tau)$. By Lemma 7.3 all its entries $\alpha_{i,j}(\tau)$ with $1\le i\le m-1$ can be expressed by the coefficients of the expansions of $t\to w_m(t,\tau)$ in the Laurent series at $t=\tau$. The entries $\alpha_{m,j}(\tau)$ do not enter the differential equation (7.23). To find an expression for these entries we will use the integral-differential equation (7.12) that can be rewritten for $\Delta = \Lambda_0(\tau)$ in the following form:

$$\frac{\partial w^{\Lambda_0(\tau)}(t,\tau_1)}{\partial \tau_1}\Big|_{\tau_1=\tau} = -m^2 w_m(t,\tau)\int^t w_m(\xi,\tau)w(\xi,\tau)d\xi \tag{7.51}$$

Using (7.20), we can obtain from here the following system of equation w.r.t. the components $w_j(t,\tau)$

$$\frac{\partial w_j(t,\tau)}{\partial \tau} + \sum_{l=1}^{m} w_l(t,\tau)\alpha_{lj}(\tau) = -m^2 w_m(t,\tau)\int^t w_m(\xi,\tau)w_j(\xi,\tau)d\xi, \;\; 1 \le j \le m \tag{7.52}$$

For given j consider the Laurent expansion of the lefthand side of (7.51), as a function of t, at $t=\tau$. By (7.25) the coefficient of $\frac{1}{t-\tau}$ in this expansion is equal to $\alpha_{m,j}(\tau)$. On the other hand, all coefficients of the appropriate expansion of the righthand side can be expressed by coefficients of expansions of $t \to w_j(t,\tau)$ and $t \to w_m(t,\tau)$ in the Laurent series at $t=\tau$. Therefore also the entries $\alpha_{m,j}(\tau)$ can be expressed by coefficients of expansions of $t \to w_i(t,\tau)$ (even with $i=j$ or m) in the Laurent series at $t=\tau$. This concludes the proof of our Lemma and also of Theorem $2'$.□

8 Appendix

In this appendix we prove the identity (7.47). We are sure that the proof presented here is far to be optimal, but this is the only one that we have at this moment.

Denote the determinant in the lefthand side of (7.47) by $L_m(k)$. Expanding this determinant w.r.t. the last column, we have

$$L_m(k) = \sum_{j=1}^{m}(-1)^{j+m}\theta_j(k)\prod_{i=1}^{j-1}\eta_i(k)\prod_{i=j+1}^{m}\zeta_i(k) + \prod_{i=1}^{m}\eta_i(k) \tag{8.1}$$

Then, substituting (7.46) in (8.1), one can easily transform $L_m(k)$ to the following form:

$$L_m(k) = \sum_{j=1}^{m+1}\frac{m(2m-j)!}{(m-j+1)!(j-1)!}\Big(k-2(m-j+1)\Big)\prod_{i=2-j}^{m-j}(k-i)\prod_{i=2m-j+2}^{2m}(k-i) \tag{8.2}$$

Note that $L_m(k)$ is a polynomial of degree $2m$, exactly as the polynomial in the righthand side of (7.47). Also for both polynomials the coefficient of leading term k^{2m} is equal to 1. Therefore in order to prove the identity (7.47) it is sufficient to prove that the polynomials in both sides of (7.47) have the same roots, or, equivalently, that $L_m(2i) = L_m(1-2i) = 0$ for all $1 \le i \le m$.

We will do this in two steps: First we will show that

$$L_m(2i) = 0 \quad 1 \le i \le m. \tag{8.3}$$

Secondly we will prove that the function

$$\overline{L_m}(k) \stackrel{def}{=} \frac{k}{k-2m} L_m(k) \tag{8.4}$$

satisfies

$$\overline{L_m}(-1-k) = \overline{L_m}(k), \tag{8.5}$$

i.e., $\overline{L_m}(k)$ is invariant under the reflection of its argument w.r.t. $-1/2$. This together with (8.3) and the fact that $\overline{L_m}(0) = 0$ (which follows directly from the definition of $\overline{L_m}(k)$) will imply that also $L_m(1-2i) = 0$ for all $1 \le i \le m$.

1. The proof of (8.3). For $1 \le j \le m+1$ denote by

$$p_{m,j}(k) = \frac{m(2m-j)!}{(m-j+1)!(j-1)!}\Big(k - 2(m-j+1)\Big) \prod_{i=2-j}^{m-j} (k-i) \prod_{i=2m-j+2}^{2m} (k-i), \tag{8.6}$$

By direct computation the following identity can be easily checked:

$$p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l) = 0, \tag{8.7}$$

where

$$0 \le l \le m-1, \quad \max\{1, 2l+1-m\} \le j \le \min\{m+1, 2l+1\}$$

In particular, applying (8.6) to $j = l+1$, we have

$$p_{m,l+1}(2m-2l) = 0 \tag{8.8}$$

By construction,

$$L_m(2m-2l) = \sum_{j=1}^{m+1} p_{m,j}(2m-2l) \tag{8.9}$$

Denote $l_1 = \max\{1, 2l+1-m\}$ and $l_2 = \min\{m+1, 2l+1\}$. Consider the following 3 cases:

1) $l_1 \le j \le l_2$. Then from (8.7) and (8.8) it follows that

$$\sum_{j=l_1}^{l_2} p_{m,j}(2m-2l) = \sum_{j=l_1}^{l} \Big(p_{m,j}(2m-2l) + p_{m,2l+2-j}(2m-2l)\Big) + p_{m,l+1}(2m-2l) = 0; \tag{8.10}$$

2) $2l+2 \le j \le m+1$. Then $2m-j+2 \le m-l \le 2m$, so from (8.6) it follows that $p_{m,j}(2m-2l) = 0$;

3) $1 \le j \le 2l - m$. Then $2 \le 2m - 2l \le m - j$ and again from (8.6) it follows that in this case $p_{m,j}(2m - 2l) = 0$.

Therefore, by (8.9), $L_m(2m - 2l) = 0$ for all $0 \le l \le m - 1$, or, equivalently, $L_m(2i) = 0$ for all $1 \le i \le m$.

2. The proof of (8.5). We will transform the expression for $L_m(k)$ to the more symmetric form. Following [5] (Chapter 1, §2) we denote

$$x^{n|h} \stackrel{def}{=} x(x+h)\dots(x+(n-1)h) \tag{8.11}$$

Then similarly to the Newton binomial identity, one easily have

$$(x+y)^{n|h} = \sum_{i=0}^{n} \begin{pmatrix} n \\ i \end{pmatrix} x^{n-i|h} y^{i|h} \tag{8.12}$$

Using the notation (8.11) one can rewrite $L_m(k)$ in the following form

$$L_m(k) = \sum_{j=1}^{m+1} \frac{m(2m-j)!}{(m-j+1)!(j-1)!}(k - 2(m-j+1))(k-m+j)^{m-1|1}(k-2m)^{j-1|1} \tag{8.13}$$

Applying (8.12), one get

$$(k-m+j)^{m-1|1} = \Big((k+1) + (j-m-1)\Big)^{m-1|1} = \sum_{i=0}^{m-1} \begin{pmatrix} m \\ i \end{pmatrix} (k+1)^{m-1-i|1}(j-m-1)^{i|1} =$$

$$\sum_{i=0}^{m-j+1} (-1)^i \begin{pmatrix} m \\ i \end{pmatrix} (k+1)^{m-1-i|1} \frac{(m-j+1)!}{(m-j-i+1)!} \tag{8.14}$$

Substituting (8.14) in (8.13) and changing the order of summation one easily obtain

$$L_m(k) = \sum_{i=0}^{m} \left(\sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2(m-j))(k-2m)^{j|1} \right) \frac{(-1)^i m!(k+1)^{m-1-i|1}}{(m-i-1)!i!}. \tag{8.15}$$

Lemma 8.1 *The following identity holds:*

$$\sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1}(k-2(m-j)) = \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i) \prod_{l=1}^{m-i-1} (k-l). \tag{8.16}$$

Proof. Using representation $k - 2(m-j) = (k-2m) + 2j$, one can split the lefthand side of (8.16) into the sum of two terms:

$$(k-2m) \sum_{j=0}^{m-i} \frac{(2m-j-1)!}{j!(m-j-i)!}(k-2m)^{j|1} + 2\sum_{j=1}^{m-i} \frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1}. \tag{8.17}$$

Since

$$(2m-j-1) = (m+i-1)!(m+i)^{m-i-j|1},$$

the first term of (8.17) can be written in the way such that one can apply the binomial identity(8.12):

$$\frac{(m+i-1)!(k-2m)}{(m-i)!}\sum_{j=0}^{m-i}\begin{pmatrix} m-i \\ j \end{pmatrix}(m+i)^{m-i-j|1}(k-2m)^{j|1} =$$

$$\frac{(m+i-1)!(k-2m)}{(m-i)!}(k-m+i)^{m-i|1}. \tag{8.18}$$

In the same way, the second term of (8.17) can be also written in the way such that one can apply the binomial identity(8.12):

$$2\sum_{j=1}^{m-i}\frac{(2m-j-1)!}{(j-1)!(m-j-i)!}(k-2m)^{j|1} = 2\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m)^{j+1|1} =$$

$$2(k-2m)\sum_{j=0}^{m-i-1}\frac{(2m-j-2)!}{(j)!(m-j-i-1)!}(k-2m+1)^{j|1} =$$

$$\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}\sum_{j=1}^{m-i-1}\begin{pmatrix} m-i-1 \\ j \end{pmatrix}(m+i)^{m-i-1-j|1}(k-2m+1)^{j|1} =$$

$$\frac{2(m+i-1)!(k-2m)}{(m-i-1)!}(k-m+i+1)^{m-i-1|1}. \tag{8.19}$$

Combining (8.18) and (8.19) together, we obtain that the lefthand side of (8.16) is equal to

$$\frac{(m+i-1)!(k-2m)}{(m-i-1)!}\Big(\frac{k-m+i}{m-i}+2\Big)(k-m+i+1)^{m-i-1|1} =$$

$$\frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)(k-m+i+1)^{m-i-1|1} = \frac{(m+i-1)!}{(m-i)!}(k-2m)(k+m-i)\prod_{l=1}^{m-i-1}(k-l)$$

that is exactly the righthand side of (8.16). This completes the proof of the lemma.□

Now substituting (8.16) to (8.15), we have the following identity:

$$L_m(k) = (k-2m)\sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\prod_{l=1}^{m-i-1}(k-l)\prod_{l=1}^{m-i}(k+l). \tag{8.20}$$

Then the function $\overline{L_m}(k)$ satisfies

$$\overline{L_m}(k) = \frac{k}{k-2m}L_m(k) = \sum_{i=0}^{m}\frac{(-1)^i m!(m+i-1)!}{i!(m-i)!(m-i-1)!}\prod_{l=1}^{m-i}(k+1-l)\prod_{l=1}^{m-i}(k+l). \tag{8.21}$$

It remains only to notice that all terms of the sum in the righthand side of (8.21) are invariant under the reflection of the argument w.r.t. $-\frac{1}{2}$ or, equivalently, under substitution $k \to -1-k$. Then the function $\overline{L_m}(k)$ is also invariant under this substitution, which proves (8.5) and therefore also (7.47).

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