A "Gauss–Bonnet Formula" for Contact Sub-Riemannian Manifolds

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Abstract

We study 3-dimensional manifolds endowed with oriented contact sub-Riemannian structures. The Euler characteristic class of the contact structure is presented as the *rotation class* of a volume preserving vector field constructed in terms of fundamental differential invariants of the sub-Riemannian metric.

1. Let M be a smooth 3-dimensional manifold. A *contact sub-Riemannian structure* is a pair $\Delta, \langle\cdot|\cdot\rangle$, where $\Delta = \{\Delta_q\}_{q\in M}$, $\Delta_q \subset T_qM$, is a contact structure on M and $\langle\cdot|\cdot\rangle = \{\langle\cdot|\cdot\rangle_q\}_{q\in M}$ is a smooth with respect to q family of Euclidean inner products

$$(v_1, v_2) \mapsto \langle v_1|v_2\rangle_q, \quad v_1, v_2 \in \Delta_q,$$

defined on Δ_q. A Lipschitzian curve $\xi : [0,1] \to M$ is called *admissible* for Δ if $\frac{d\xi(t)}{dt} \in \Delta_{\xi(t)}$ for almost all $t \in [0,1]$. The *length* of an admissible curve ξ is the integral $\int\limits_0^1 |\frac{d\xi}{dt}|\, dt$, where $|v| = \sqrt{\langle v|v\rangle_q} \quad \forall v \in T_qM$. The infimum of the lengths of admissible curves connecting two points is the *Carnot–Caratheodory distance* between these points.

An important class of the sub-Riemannian structures is provided by magnetic fields on Riemannian surfaces. In this case M is the total space of a principal $\mathbb{U}(1)$-bundle over a Riemannian surface N and Δ is a connection on the principal bundle. In other words, Δ is a transversal to fibers $\mathbb{U}(1)$-invariant rank 2 distribution on M. The distribution is contact if and only if the curvature of the connection doesn't vanish. The inner product of a pair of vectors in Δ_q equals the scalar product of their projections in T_qN.

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The length minimization problem for admissible curves is equivalent to the least action principle for the charged particles in the magnetic field (see [5] for details). The special case of the *constant* magnetic field (when the curvature of the connection is the area form multiplied by a constant) is locally equivalent to the classical Dido isoperimetric problem on the Riemannian surface.

The local structure of the Carnot–Caratheodory metric was studied in detail in the papers [1, 2, 3, 4]; it was shown that this structure is controlled by some fundamental differential invariants. In this note the same local invariants serve to express a global one: the Euler characteristic class of Δ.

2. We assume that Δ is an oriented contact structure. Then there exists a unique contact form ω on M such that Δ is the annihilator of ω, $\Delta_q = \omega_q^{\perp}$ $\forall q \in M$ and the form $d\omega|_{\Delta_q}$ coincides with the area form on the oriented Euclidean plane Δ_q. Let X_0 be the characteristic (or Reeb) vector field of ω; it is defined by the relations $X_0 \rfloor d\omega = 0$, $X_0 \rfloor \omega = 1$.

Let $\delta(q_0, q_1)$ be the Carnot–Caratheodory distance between q_0 and q_1 and

$$\mathcal{C}_{q_0} = \{q_1 \in M : \text{ the function } q \mapsto \delta(q_0, q) \text{ is not } C^1 \text{ at } q_1\}.$$

One can show that the line $\mathbb{R}X_0(q)$ is the tangent cone[1] to the $\mathcal{C}_q$ at q, $\mathbb{R}X_0(q) = T_q\mathcal{C}_q$.

The curve $t \mapsto e^{tX_0}(q)$ is transversal to the distribution Δ. Let $N_q \subset M$ be (a germ of) a 2-dimentional submanifold such that $\Delta_q = T_qN$; then a neighborhood of q is sliced by the submanifolds $e^{tX_0}(N_q)$. One of the basic invariants of the sub-Riemannian structure arises from the asymptotics of the distance between q and $e^{tX_0}(N_q) \cap \mathcal{C}_q)$ as $t \longrightarrow 0$:

$$\delta(q, e^{tX_0}(N_q) \cap \mathcal{C}_q) = (2 - \kappa(q)t)|\pi t|^{1/2} + O(t^2),$$

where κ is a smooth function on M.

Let $X_1(q), X_2(q)$ be an orthonormal frame in Δ_q. The explicit expression of κ in terms of the moving frame X_0, X_1, X_2 is as follows:

$$\kappa = X_1 c_{12}^2 - X_2 c_{12}^1 - (c_{12}^1)^2 - (c_{12}^2)^2 + \frac{1}{2}(c_{02}^1 - c_{01}^2),$$

where $[X_i, X_j] = \sum_k c_{ij}^k X_k$.

[1] There are many definitions of the tangent cone to the closed set; all of them provide one and the same cone in this particular case.

In the special case of the constant magnetic field on the Riemannian surface (see Sec. 1) X_0 is a generator of the structural group of the principle bundle, κ is constant on the fibers of the bundle and is actually the pullback of the Gaussian curvature of the Riemannian surface.

3. Now we take the dual object to the sub-Riemannian structure, the Hamiltonian h on the cotangent bundle T^*M:

$$h(\lambda) = \frac{1}{2}(\max\{\langle\lambda, v\rangle : v \in \Delta_q,\ |v| = 1\})^2, \quad \lambda \in T_q^*M,\ q \in M.$$

This Hamiltonian serves to describe sub-Riemannian geodesics, i.e. admissible curves whose small pieces are length minimizers. It follows from the Pontryagin Maximum Principle that geodesics are exactly projections to M of the trajectories of the Hamiltonian system in T^*M associated with h.

Let $u_i(\lambda) = \langle\lambda, X_i(q)\rangle$, $i = 1, 2$; then $h(\lambda) = \frac{1}{2}\left(u_1(\lambda)^2 + u_2(\lambda)^2\right)$, $\lambda \in T_q^*M$. The Hamiltonian keeps all the information on the sub-Riemannian structure: both Δ and the inner product are easily recovered from h.

We denote: $u_0(\lambda) = \langle\lambda, X_0(q)\rangle$, $\lambda \in T_q^*M$, $q \in M$, and $\Delta^* = u^{-1}(0)$. Then Δ^* is a rank 2 linear subbundle of T^*M with the fibers $\Delta_q^* = \Delta^* \cap T_q^*M$. Obviously, $(\lambda, \xi) \mapsto \langle\lambda, \xi\rangle$, $\lambda \in \Delta_q^*$, $\xi \in \Delta_q$, is a nondegenerate pairing. Moreover, quadratic form $2h|_{\Delta_q^*}$ defines the Euclidean structure on Δ_q^* dual to the given Euclidean structure on Δ_q.

We'll deal with homogeneous polynomials on the plane Δ_q^*. The group $\mathrm{SO}(\Delta_q^*)$ acts on the polynomials by the changing of variables. Irreducible components of this (real) action are 2-dimensional spaces of polynomials having the following expression in polar coordinates (r, θ):

$$span\{r^n \cos(k\theta),\ r^n \sin(k\theta)\}, \tag{1}$$

where $n - k$ is a nonnegative even number. Note that $r^2 = 2h|_{\Delta_q^*}$. So any homogeneous degree n polynomial ϕ has a unique presentation as a sum of isotopic components: $\phi = \sum\limits_{i=0}^{[n/2]} \phi^{n-2i}$, where ϕ^k belongs to the space (1). This presentation is actually equivalent to the Fourier expansion of the restriction of ϕ to the unit circle.

4. Recall that h, u_0 are functions on the cotangent bundle T^*M, where restrictions of h to the fibers T_q^*M are quadratic forms and restrictions of u_0 to the fibers are linear forms. Hence the Poisson bracket $\{h, u_0\}$ is quadratic and the double Poisson bracket $\{h, \{h, u_0\}\}$ is cubic on the fibers. Let $\phi_q = \{h, \{h, u_0\}\}|_{\Delta_q^*}$; then $\phi_q = \phi_q^1 + \phi_q^3$, where ϕ_q^1 is the product of $h|_{\Delta_q^*}$ and a linear form on Δ_q^*, according to (1). In other words, $\phi_q^1(\lambda) = h(\lambda)\langle\lambda, f(q)\rangle$ for some $f(q) \in \Delta_q$.

We thus obtain an intrinsically defined vector field f with values in Δ in addition to the transversal to Δ field X_0.

Theorem 1 *The 2-form on M*

$$\left(\frac{\kappa}{2\pi}X_0 - \frac{1}{\pi}f\right)\rfloor\, \omega \wedge d\omega \tag{2}$$

is closed and represents the Euler characteristic class of the oriented linear bundle Δ.

Remark 1. The statement of the theorem can be also formulated as follows: The flow generated by the field

$$\left(\frac{\kappa}{2\pi}X_0 - \frac{1}{\pi}f\right) \tag{3}$$

preserves the volume form $\omega \wedge d\omega$ and the *rotation class* (see [6]) of the field (3) is equal to the Euler class of Δ.

Remark 2. In the case of the principal bundle $M \xrightarrow{\mathbb{U}(1)} N$ and the sub-Riemannian structure defined by a *constant* magnetic field (see sec.1) we have $\{h, u_0\} = 0$ and hence $f = 0$. Then (2) takes the form : $\frac{\kappa}{2\pi}X_0\rfloor\omega\wedge d\omega = \frac{\kappa}{2\pi}d\omega$. Moreover, $d\omega$ and κ are the pullbacks of the area form and of the Gaussian curvature on N so that the form (2) turns into the pullback of the Gauss–Bonnet form on N.

The proof of Theorem 1 consists of a calculation with moving frames in T^*M. The idea is to construct an appropriate linear connection on the bundle $\Delta^* \subset T^*M$ via the Hamiltonian vector fields associated with h and u_0 and a vertical vector field generating rotations of the fibers Δ_q^*. The form (2) is the curvature form of the correspondent linear connection.

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