Feedback–invariant Optimal Control Theory and Differential Geometry, II. Jacobi Curves for Singular Extremals

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Abstract

This is the second article in the series opened by the paper [4]. Jacobi curves were defined, computed, and studied in that paper for regular extremals of smooth control systems. Here we do the same for singular extremals. The last section contains a feedback classification and normal forms of generic single–input affine in control systems on a 3-dimensional manifold.

Introduction

This paper is a continuation of [4]. Jacobi curves were defined, computed, and studied in [4] for regular extremals of smooth control systems. Here we do the same for singular extremals.

The points of the Jacobi curves are the $\mathcal{L}$-derivatives of endpoint mappings. The notion of $\mathcal{L}$-derivative was introduced in [4]; in section 1 of the present paper we prove a general existence theorem for the $\mathcal{L}$-derivatives of smooth mappings and indicate a way to compute them. In section 2 we actually compute the $\mathcal{L}$-derivatives of the endpoint mappings and the Jacobi curves for a wide class of control systems and their extremals. In section 3 we apply the theory to a low dimensional example: we define the curvature for 3-dimensional control systems of the form $\dot{x} = g^0(x) + ug^1(x)$ and give a local classification of such systems under some nondegeneracy conditions.

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The Jacobi curves studied here and in [4] are determined by the $\mathcal{L}$-derivatives and are automatically feedback–invariant. To prevent a possible misunderstanding I have to mention that in all earlier papers we used the term "Jacobi curve" for similar but not the same (and not feedback–invariant) objects. We, however, extensively apply results and techniques of that earlier papers for the computing and studying this new feedback–invariant Jacobi curves.

1 The Existence of the $\mathcal{L}$-derivative

We use definitions and notations from [4, Sec. 1]; see also Appendix for the definition and main properties of the Maslov-type index ind_Π. Let (u,λ) be a Lagrangian point of the mapping $f : U \to M$ and let N be a germ of a submanifold in U at u, hence (u,λ) is a Lagrangian point of $f|_N$. If N is finite–dimensional, then $\mathcal{L}_{(u,\lambda)}(f|_N)$ is a Lagrangian subspace. Denote by $\mathcal{N}$ the set of all such germs partially ordered by inclusion. Then $\left\{\mathcal{L}_{(u,\lambda)}(f|_N)\right\}_{N\in\mathcal{N}}$ is a generalized sequence of points of the Lagrangian Grassmannian $L(T_\lambda(T^*M))$.

Theorem 1.1 *If the negative or the positive inertia index of $Hess_{(u,\lambda)}f$ is finite, then there exists the limit $\mathcal{N}$-$lim\,\mathcal{L}_{(u,\lambda)}(f|_N)$.*

Proof. We fix local coordinates in U, M and hence in T^*M. In these coordinates, $f(u)$ is the origin 0 in $\mathbb{R}^n$, $\lambda = (p,0)$, where $p \in \mathbb{R}^{n*}$. Let $\Pi = \{(\xi,0) : \xi \in \mathbb{R}^{n*}\}$ be the fiber of T^*M at $f(u)$ identified with the tangent space to the fiber. Set $Qv = pf''_u v$, $\bar{Q}(v) = \langle Qv, v\rangle$, $v \in T_uU$. We may deal only with the negative inertia index and may assume without lack of generality that $\ker Q \cap \ker f'_u = 0$.

Lemma 1.1 *Let $N_1 \subset N_2$ be germs at u of finite–dimensional submanifolds in U, $V_i = T_uN_i$, $V_i^0 = V_i \cap \ker f'_u$ $i = 1,2$. If $\mathrm{rank}\, f'_u|_{V_1} = \mathrm{rank}\, f'_u|_{V_2}$ and $\ker \bar{Q}|_{V_1^0} = \ker \bar{Q}|_{V_2^0}$, $i = 1,2$, then*

$$\mathrm{ind}_-\,\bar{Q}|_{V_2^0} \geq \mathrm{ind}_-\,\bar{Q}|_{V_1^0} + \mathrm{ind}_\Pi(\mathcal{L}(f|_{N_1}), \mathcal{L}(f|_{N_2})).$$

Proof. The assumption on kernels of $\bar{Q}|_{V_i^0}$ implies

$$\mathrm{ind}_-\,\bar{Q}|_{V_2^0} = \mathrm{ind}_-\,\bar{Q}|_{V_1^0} + \mathrm{ind}_-\,\bar{Q}|_{V_2\cap W_1},$$

where $W_1 = \{w : \langle Qw + \xi f'_u, V_1\rangle = 0,\ f'_u w = 0, \text{for some } \xi \in R^{n*}\}$.

On the other hand,

$$\operatorname{ind}_\Pi(\mathcal{L}(f|_{N_1}),\mathcal{L}(f|_{N_2})) = \operatorname{ind}_- S +$$

$$+\frac{1}{2}\left(\dim(\mathcal{L}(f|_{N_1})\cap\Pi)+\dim(\mathcal{L}(f|_{N_2})\cap\Pi)\right)-\dim\left(\bigcap_{i=1}^{2}\mathcal{L}(f|_{N_i})\cap\Pi\right),$$

where S is a standard quadratic form defined on $\left((\mathcal{L}(f|_{N_1}+\mathcal{L}(f|_{N_2})\right)\cap\Pi$. (see Append.). Recall that

$$\mathcal{L}(f|_{N_i}) = \{(\xi_i, f'_u v_i) : \langle Qv_i+\xi_i f'_u, V_i\rangle = 0,\ v_i\in V_i\}.$$

The assumption on kernels of $\bar{Q}|_{V_i^0}$ implies that $\mathcal{L}(f|_{N_1})\cap\Pi = \mathcal{L}(f|_{N_2})\cap\Pi$. Hence $\operatorname{ind}_\Pi(\mathcal{L}(f|_{N_1}),\mathcal{L}(f|_{N_2})) = \operatorname{ind} S$. The quadratic form S is computed in terms of the symplectic structure σ. Let $(\xi_i, f'_u v_i)\in\mathcal{L}(f|_{N_i})$, $f'_u(v_1+v_2)=0$, $\xi=\xi_1+\xi_2$. Then

$$S(\xi) = \sigma((\xi_1, f'_u v_1),(\xi_2, f'_u v_2)) = \xi_1 f'_u v_2 - \xi_2 f'_u v_1 = -\xi f'_u v_1.$$

We set $w = v_1+v_2$, then $w\in V_2\cap W_1$ and we obtain

$$\bar{Q}(w) = \langle Qw, w\rangle = \langle Q(v_1+v_2), v_1+v_2\rangle = \langle Qv_1, v_1+v_2\rangle =$$

$$= \langle Q(v_1+v_2), v_1\rangle = \langle Qw, v_1\rangle = -\xi f'_u v_1 = S(\xi).$$

Hence S is equivalent to the restriction of $\bar{Q}$ to the subspace in $V_2\cap W_1$ and $\operatorname{ind}_- S\le\operatorname{ind}_-\bar{Q}|_{V_2\cap W_1}$. $\square$

Lemma 1.2 *If* $\operatorname{ind}_- Q|_{\ker f'_u} < \infty$, *then there exists a finite–dimensional subspaces* $V_o\subset T_uU$, $V_o^0 = V_o\cap\ker f'_0$ *such that* $\operatorname{rank} f'_u = \operatorname{rank} f'_u|_{V_o}$, $\ker\bar{Q}|_W = \ker\bar{Q}|_{V_o^0}$, *and* $\operatorname{ind}_-\bar{Q}|_W = \operatorname{ind}_-\bar{Q}|_{V_o^0}$ *for any* W *satisfying the inclusions* $V_o^0\subset W\subset\ker f'_u$.

The proof is left to the reader. $\square$

Let $V = T_uN$, then $\mathcal{L}(f|_N)$ depends on V rather than on N. We introduce a shorter notation $\Lambda_V = \mathcal{L}(f|_N)$.

For any $V\supset V_o$, we have $\Pi\cap\Lambda_V = \Pi\cap\Lambda_{V_o}$. We introduce a reduced symplectic space $\Sigma = (\Pi\cap\Lambda_{V_o})^\angle/(\Pi\cap\Lambda_{V_o})$, $\dim\Sigma = 2m\le 2n$. Then $\forall V_2\supset V_1\supset V_o$, Λ_{V_i} belongs to the Lagrange Grassmannian $L(\Sigma)$,

$$\operatorname{ind}_\Pi(\Lambda_{V_1},\Lambda_{V_2}) = 0,$$

and Λ_{V_i} are transversal to Π in Σ.

Take $\Delta \in L(\Sigma)$ such that $\mathrm{ind}_\Pi(\Lambda_{V_o}, \Delta) = m$. The triangle inequality (see Appendix)

$$\mathrm{ind}_\Pi(\Lambda_{V_o}, \Delta) \le \mathrm{ind}_\Pi(\Lambda_{V_o}, \Lambda_V) + \mathrm{ind}_\Pi(\Lambda_V, \Delta)$$

implies that $\mathrm{ind}_\Pi(\Lambda_V, \Delta) = m \; \forall V \supset V_o$. In particular, Λ_V is transversal to Δ in $L(\Sigma)$.

Identifying the chart $\Delta^\pitchfork$ in $L(\Sigma)$ with the affine space over the linear space of quadratic forms on $\Delta^* \cong \Sigma/\Delta$ we obtain $\mathrm{ind}_\Pi(\Lambda_V, \Delta) = \mathrm{ind}(\Pi - \Lambda_V)$ (the inertia index of the quadratic form in the right–hand side). In particular, $\Pi - \Lambda_V$ is identified with a negative definite quadratic form. Finally, we make the affine space $\Delta^\pitchfork$ linear fixing Π as the origin. Then Λ_V is identified with a positive definite quadratic form.

We are going to show that $\mathrm{ind}_\Pi(\Lambda_{V_1}, \Lambda_{V_2}) = 0$. It follows from (4.1) that $\mu(\Lambda_{V_1}, \Pi, \Delta) = \mu(\Lambda_{V_1}, \Pi, \Delta) = -m$. Mow the chain rule for μ applied to the quadruple $\Lambda_{V_1}, \Lambda_{V_2}, \Pi, \Delta$ implies $\mu(\Lambda_{V_1}, \Delta, \Lambda_{V_2}) = \mu(\Lambda_{V_1}, \Pi, \Lambda_{V_2})$. Recall that Maslov index μ and the dimensions of the intersections (by pairs and all three together) form a complete system of invariants of the triples of Lagrangian subspaces. Since $\Lambda_{V_1}, \Lambda_{V_2}$ are transversal to both Π and Δ, we obtain that $\mathrm{ind}_\Delta(\Lambda_{V_1}, \Lambda_{V_2}) = \mathrm{ind}_\Pi(\Lambda_{V_1}, \Lambda_{V_2})$. Hence $\mathrm{ind}_\Pi(\Lambda_{V_1}, \Lambda_{V_2}) = 0$; this relation implies the existence of a monotonically decreasing curve in $\Delta^\pitchfork$ connecting Λ_{V_1} with Λ_{V_2}. Hence $\Lambda_{V_2} \le \Lambda_{V_1}$ in the space of quadratic forms.

Remark 1.1 The signs of quadratic forms under consideration and the monotonicity types of curves in the Lagrange Grassmannian depend on the general sign agreement which varies from paper to paper. We follow the agreement of the paper [4] and also identify the coset $(z + \Delta) \in \Sigma/\Delta$ with the linear functional $\sigma(z, \cdot)$ on Δ.

So Λ_V, $V \supset V_o$, form a monotonically decreasing generalized sequence of positive definite quadratic forms. The sequence must have the limit which is a nonnegative quadratic form or, in other words, an element of $\Delta^\pitchfork \subset L(\Sigma)$. $\square$

Let a be a smooth function on M, $da \subset T^*M$ be the differential of a, which is a Lagrangian submanifold of T^*M. Let $\lambda \in da$, we set $\Pi_\lambda(a) = T_\lambda(da)$. Then u is a critical point of the scalar function $a \circ f$ and the Hessian $Hess_u(a \circ f)$ is a well–defined quadratic form on T_uU, $Hess_u(a \circ f)(v) = \langle (a \circ f)''_u v, v \rangle$. One can prove the following modification of Lemma 1 in the same way as that lemma.

Lemma 1.3 *Let $N_1 \subset N_2$ be germs at u of finite–dimensional submanifolds in U, $V_i = T_u N_i$, $V_i^0 = V_i \cap \ker f'_u$ $i = 1, 2$. If* $\operatorname{rank} f'_u|_{V_1} = \operatorname{rank} f'_u|_{V_2}$ *and* $\ker(Hess_u(a \circ f)|_{V_1^0}) = \ker(Hess_u(a \circ f)|_{V_2^0})$, *$i = 1, 2$, then*

$$\operatorname{ind}_- Hess_u(a \circ f)|_{V_2^0} \geq \operatorname{ind}_- Hess_u(a \circ f)\ |_{V_1^0} + \operatorname{ind}_{\Pi_\lambda(a)}(\mathcal{L}(f|_{N_1}), \mathcal{L}(f|_{N_2})).$$

□

The limit from Theorem 1.1 is the precise definition of the $\mathcal{L}$-derivative of f at the Lagrangian point (u, λ). But it is not enough to prove the existence of the limit, we must compute it. Introducing local coordinates, we may assume that U is a Banach space and u its origin. Let $U_0 \subset U$ be an arbitrary linear subspace which is dense in U, and $\mathcal{N}_0$ be the set of all finite–dimensional subspaces $N_0 \subset U_0$, partially ordered by inclusion. Thus, $\mathcal{N}_0 \subset \mathcal{N}$. The following assertion is an essential addition to Theorem 1.1, making possible to explicitly compute the limit indicated in the theorem.

Proposition 1.1 *Under the hypothesis of Theorem 1.1 the following equality holds:*

$$\mathcal{N}_0\text{-}\lim \mathcal{L}_{(u,\lambda)}(f|_{N_0}) = \mathcal{N}\text{-}\lim \mathcal{L}_{(u,\lambda)}(f|_N).$$

The proof is based on Lemma 1.3. We didn't use the completeness of $T_u U$ in the proof of Theorem 1.1, hence the limit in the left–hand side does exist. Suppose

$$\Lambda_0 = \mathcal{N}_0\text{-}\lim \mathcal{L}_{(u,\lambda)}(f|_{N_0}), \quad \Lambda = \mathcal{N}\text{-}\lim \mathcal{L}_{(u,\lambda)}(f|_N).$$

Take an arbitrary $\Delta \in \Pi^{\pitchfork}$ and the function a such that $T_\lambda(da) = \Delta$. Using the fact that $\operatorname{ind}_-(Hess_u(a \circ f)|_{U_0}) = \operatorname{ind}_- Hess_u(a \circ f)$ and applying Lemma 1.3 one can prove that $\operatorname{ind}_\Delta(\Lambda_0, \Lambda) = 0$. This is possible for all Δ only if $\Lambda_0 = \Lambda$ (in fact, it would be enough to consider any everywhere dense subset of "points" Δ in the Lagrange Grassmannian). □

Remark 1.2 The $\mathcal{L}$-derivative Λ contains the subspace $L = \{(\xi, f'_u v) : Qv + \xi f'_u = 0,\ v \in U\}$ and coincides with L if $Q(U)$ is closed in U. In particular, $\dim L \leq n$. It is often possible to endow the space U with a natural weaker topology preserving the continuity of Q and f'_u and such that $Q(\bar{U})$ is closed in $\bar{U}$, where $\bar{U}$ is the completion of U in the new topology. Then Λ coincides with $\bar{L} = \{(\xi, f'_u v) : Qv + \xi f'_u = 0,\ v \in \bar{U}\}$. It follows from Proposition 1.1 applied to the space $\bar{U}$ with the everywhere dense subspace U and the mapping $(f'_u, \bar{Q})$ from $\bar{U}$ into $\mathbb{R}^n = \{p\}^\perp \times \mathbb{R}$. The equality $\Lambda = \bar{L}$ still holds if $Q(\bar{U})$ is not closed but $\bar{L}$ is n-dimensional.

2 Computing the Jacobi Curves

We use definitions and notations from [4, Sec. 2] and we deal with a given extremal $\tau \mapsto (\xi(\tau), \lambda_\tau)$, where the "control" ξ is presented in the form $\xi(\tau) = (\tilde{u}(\tau), \tilde{x}(\tau))$, $\frac{d}{d\tau}\tilde{x} = f_\tau(\tilde{x}(\tau), \tilde{u}(\tau))$, $0 \le \tau \le t$. Moreover, we assume that isomorphisms $T_{\tilde{u}(\tau)}U \approx \mathbb{R}^r$ are fixed; it makes $\frac{\partial f_\tau}{\partial u}(\tilde{x}(\tau), \tilde{u}(\tau))$ a linear mapping from $\mathbb{R}^r$ into $T_{\tilde{x}(\tau)}M$. We also assume that $\frac{\partial f_\tau}{\partial u}(\tilde{x}(\tau), \tilde{u}(\tau))$ is Lipschitzian with respect to τ.

Recall that the extremal is regular if and only if $\frac{\partial^2(\lambda_\tau f_\tau)}{\partial u^2}(\tilde{x}(\tau), \tilde{u}(\tau))$ is a nondegenerate quadratic form for all $\tau \in [0, t]$. We assume that $\frac{\partial^2(\lambda_\tau f_\tau)}{\partial u^2}(\tilde{x}(\tau), \tilde{u}(\tau)) = 0\ \forall \tau \in [0, t]$, so that $\tau \mapsto (\xi(\tau), \lambda_\tau)$ is a totally singular extremal. We set

$$g_\tau(\lambda, u) = \lambda(\overrightarrow{\exp} \int_t^\tau ad\, \tilde{f}_\theta\, d\theta (f_\tau(\cdot, u) - \tilde{f}_\tau)),$$

the same notation as in [4, Sec. 3]. In particular,

$$g_\tau(\lambda_t, u) = \lambda_\tau(f_\tau(\tilde{x}(\tau), u) - f_\tau(\tilde{x}(\tau), \tilde{u}(\tau)).$$

We also denote $X_\tau = \frac{\partial \vec{g}_\tau}{\partial u}(\lambda_t, \tilde{u}(\tau))$; then $X_\tau : \mathbb{R}^r \to T_{\tilde{x}(\tau)}M$ is a Lipschitzian with respect to $\tau \in [0, t]$ family of linear mappings.

Let $F_{0,t} = F_t|^t_{\Omega_{x_0}} : \xi \mapsto pr \circ \xi(t) = x(t)$ be the endpoint mapping, which is defined on the space of admissible controls subject to the condition $x(0) = pr \circ \xi(0) = x_0$. It follows from proposition 3.1 and equality (3.2) of [4] that

$$\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t}) = \{\eta_t \in T_{\lambda_t}(T^*M) : \exists \eta_\tau \in T_{\lambda_t}(T^*M), v(\tau) \in \mathbb{R}^r\ (0 \le \tau \le t),$$

$$\text{s. t. } \dot{\eta} = X_\tau v(\tau), \sigma(X_\tau \cdot, \eta_\tau) = 0, \eta_0 \in T_{\lambda_t}(T^*_{\tilde{x}(\tau)}M)\},$$

where $\tau \mapsto \eta_\tau$ is Lipschitzian and $\tau \mapsto v_\tau$ is bounded measurable. $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ is an isotropic subspace of $T_{\lambda_t}(T^*M)$; recall that $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ is contained in the $\mathcal{L}$-derivative of $F_{0,t}$ at $(\xi, \lambda_t$, if the $\mathcal{L}$-derivative does exist. It is worth to note that, contrary to the regular case, the dimension of $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ is usually stricly less than n, so that $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ is not a Lagrangian subspace and hence it is not the $\mathcal{L}$-derivative. Indeed, the vectors $X_\tau v$ are skew-orthogonal to $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ though these vectors do not belong to $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ except of very degenerate cases.

Proposition 2.1 ([1, 2, 5]) *If the negative inertia index of the quadratic form*
$Hess_{(\xi,\lambda_t)}F_{0,t}$ *is finite, i.e.* $\mathrm{ind}_-\, Hess_{(\xi,\lambda_t)}F_{0,t} < \infty$, *then*

$$\sigma(X_\tau v_1, X_\tau v_2) = 0, \quad \sigma(\dot{X}_\tau v_1, X_\tau v_1) \geq 0, \quad \forall v_i \in \mathbb{R}^r, \tau \in [0,t].$$

If, additionally, $\sigma(\dot{X}_\tau v, X_\tau v) \geq \alpha|v|^2$ *for some* $\alpha > 0$*, any* $v \in \mathbb{R}^r$*, and almost all* $\tau \in [0,t]$*, then indeed* $\mathrm{ind}_-\, Hess_{(\xi,\lambda_t)}F_{0,t} < \infty$.

The identity from the above proposition is called the Goh condition while the inequality is called the (first) generalized Legendre condition.

The finiteness of $\mathrm{ind}_-\, Hess_{(\xi,\lambda_t)}F_{0,t}$ provides the existence of the $\mathcal{L}$-derivative of $F_{0,t}$ at (ξ, λ_t). Our nearest goal is to compute the $\mathcal{L}$-derivative under conditions

$$\sigma(X_\tau v_1, X_\tau v_2) = 0, \quad \sigma(\dot{X}_\tau v, X_\tau v) \geq \alpha|v|^2, \tag{2.1}$$

i.e. under the Goh condition and the strengthened (first) generalized Legendre condition. Differentiating the identity $\sigma(X_\tau v_1, X_\tau v_2) = 0$ with respect to τ we obtain that $\sigma(\dot{X}_\tau v_1, X_\tau v_2)$ is a symmetric bilinear form of v_1, v_2. We define the linear mapping $b_\tau : \mathbb{R}^r \to \mathbb{R}^{r*}$ by the identity $\langle b_\tau v_1, v_2 \rangle = \sigma(\dot{X}_\tau v_1, X_\tau v_2)$. Then b_τ is selfadjoint and invertible. In particular, $b_\tau^{-1} : \mathbb{R}^{r*} \to \mathbb{R}^r$ is a selfadjoint mapping and it may be identified with a quadratic form on $\mathbb{R}^{r*}$; we denote this quadratic form by $b_\tau^{-1}(\cdot)$ so that $b_\tau^{-1}(w) \stackrel{def}{=} \langle w, b_\tau^{-1} w \rangle$, $w \in \mathbb{R}^{r*}$.

Proposition 2.2 *Under conditions (2.1), the* $\mathcal{L}$*-derivative of* $F_{0,t}$ *at* (ξ, λ_t) *is a subspace* $\Lambda_t \subset T_{\lambda_t}(T^*M)$ *which is the sum of the subspace* $\{X_t v : v \in \mathbb{R}^r\}$ *and the subspace consisting of the values at* t *of the solutions* $\tau \mapsto \eta(\tau)$ *of the linear Hamiltonian system on* $T_{\lambda_t}(T^*M)$ *associated to the nonstationary quadratic Hamiltonian*

$$q_\tau(\eta) = -\frac{1}{2} b_\tau^{-1}(\sigma(\dot{X}_\tau \cdot, \eta))$$

with the initial conditions $\eta(0) \in T_{\lambda_t}(T^*_{\tilde{x}(t)}M)$, $\sigma(\eta(0), X_0 v) = 0 \ \forall v \in \mathbb{R}^r$.

Proof. Let us first show that Λ_t is a Lagrangian subspace. We have $\vec{q}_\tau(\eta) = \dot{X}_\tau b_\tau^{-1} \sigma(\dot{X}_\tau \cdot, \eta)$. In particular,

$$\vec{q}_\tau(X_\tau v) = \dot{X}_\tau b_\tau^{-1} \sigma(\dot{X}_\tau \cdot, X_\tau v) = \dot{X}_\tau b_\tau^{-1} b_\tau v = \dot{X}_\tau v.$$

Hence $\tau \mapsto X_\tau v$ is a solution of the Hamiltonian system $\dot\eta = \vec{q_\tau}(\eta)$. Let $\Phi_\tau \in Sp(T_{\lambda_t}(T^*M))$, $\tau \in [0,t]$, be a linear symplectic flow generated by this system, $\Phi_0 = id$. We also denote $\Gamma_\tau = \{X_\tau v : v \in \mathbb{R}^r\}$, $\Pi_t = T_{\lambda_t}(T^*_{\tilde{x}(t)}M)$. Then Γ_τ is a family of isotropic subspaces and

$$\Lambda_t = \Phi(\Pi_t \cap \Gamma_0^\angle) + \Gamma_t = \Phi_t(\Pi_t) \cap \Gamma_t^\angle + \Gamma_t = \Phi(\Pi_t \cap \Gamma_0^\angle + \Gamma_0).$$

In other words, $\Lambda_t = \Phi(\Pi_t^{\Gamma_0} = \Phi(\Pi_t)^{\Gamma_t}$ (see Appendix for the notation).

So Λ_t is a Lagrangian subspace. In order to prove that Λ_t is the $\mathcal{L}$-derivative we'll use the method described in Remark 1.2. Let us recall the construction of $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$. This subspace consists of the vectors $(\eta_0 + \int\limits_0^t X_\tau v(\tau)\,d\tau)$, where $\eta_0 \in \Pi_t$, the vector function $v(\cdot)$ belongs to $L^r_\infty[0,t]$ and satisfies the equation

$$\sigma(\eta_0 + \int\limits_0^\tau X_\theta v(\theta)\,d\theta, X_\tau \cdot) = 0, \quad 0 \le \tau \le t. \tag{2.2}$$

The operator $v(\cdot) \mapsto \sigma((\eta_0 + \int\limits_0^{\cdot} X_\theta v(\theta)\,d\theta, X.)$ is continuous in a much weaker topology than the topology of $L^r_\infty[0,t]$; namely, this operator is continuous in the topology of the Sobolev space $H^r_{-1}[0,t]$. More precisely, the norm $\|\cdot\|_{-1}$ of $H^r_{-1}[0,t]$ has a form $\|v\|_{-1} = \left(|w_t|^2 + \int\limits_0^t w_\tau^2\,d\tau\right)^{\frac{1}{2}}$, where $w_\tau = \int\limits_0^t v(\tau)\,d\tau$.

The completion of $L^r_\infty[0,t]$ in the norm $\|\cdot\|_{-1}$ consists of the all pairs $(w_t, w(\cdot))$, $w_t \in \mathbb{R}^r$, $w(\cdot) \in L^r_2[0,t]$. Integrating by part and applying the Goh condition we obtain that the extension of equation (2.2) to the space $H^r_{-1}[0,t]$ has a form

$$\sigma(\eta_0 - \int\limits_0^\tau \dot X_\theta w(\theta)\,d\theta, X_\tau \cdot) = 0, \quad 0 \le \tau \le t,$$

$$\eta_0 \in \Pi_t, \quad , w_t \in \mathbb{R}^r, \quad w(\cdot) \in L^r_2[0,t]; \tag{2.3}$$

the corresponding extension of $\mathcal{L}_{(\xi,\lambda_t)}(F_{0,t})$ consists of the vectors $\eta_0 + X_t w_t - \int\limits_0^t \dot X_\tau w(\tau)\,d\tau$, where $\eta_0, w_t, w(\cdot)$ satisfy equation (2.3). Set

$$\eta_\tau = \eta_0 + X_\tau w_t - \int\limits_0^\tau \dot X_\theta w(\theta)\,d\theta.$$

If the data satisfy (2.3), then $\eta(0) \in (\Pi \cap \Gamma_0^\angle + \Gamma_0) = \Pi_t^{\Gamma_0}$. We have to show that

$$\Lambda_t = \{\eta(t) : \sigma(\eta(\tau), X_\tau \cdot) \equiv 0, \dot\eta(\tau) = X_\tau w(\tau), w \in L_2^r[0,t], \eta(0) \in \Pi_t^{\Gamma_0}\}.$$

Differentiating the equality $\sigma(\eta(\tau), X_\tau \cdot) = 0$ gives the identity

$$-\sigma(\dot X_\tau w(\tau), X_\tau \cdot) + \sigma(\eta(\tau), \dot X_\tau \cdot) = 0,$$

which can be rewritten in the form $b_\tau w(\tau) = \sigma(\eta(\tau), \dot X_\tau \cdot)$, or

$$w(t) = b_\tau^{-1} \sigma(\eta(\tau), \dot X_\tau \cdot).$$

Thus

$$\dot\eta = -\dot X_\tau b_\tau^{-1} \sigma(\eta(\tau), \dot X_\tau \cdot) = \vec q_\tau(\eta)$$

and we are done. $\square$

We stay with the notations of the paper [4] as long as it is possible. In particular, $F_{\tau,t} = F_t|_{\Omega^{t-\tau}_{\tilde x(\tau)}}$ (see [4, Sec. 4]). The difference with the mentioned paper is the singularity of the extremal (ξ, λ_t). Because of this, the definition of the quadratic Hamiltonian q_τ must be modified. The definition of the Jacobi curve must be modified as well. Suppose that (ξ, λ_t) satisfies relations (2.1), and let $\Lambda_{\tau,t}$ be the $\mathcal{L}$-derivative of $F_{\tau,t}$ at the point $(\xi|_{[\tau,t]}, \lambda_t)$. We may apply Proposition 2.2 to construct $\Lambda_{\tau,t}$ since $(\xi|_{[\tau,t]}, \lambda_t)$ satisfies the Goh condition and the strengthened generalized Legendre condition.

Let $\Phi_{\tau,s} \in Sp(T_{\lambda_t}(T^*M))$ be the family of linear symplectic transformations defined by the relations

$$\frac{\partial}{\partial s}\Phi_{\tau,s} = \vec q_s \Phi_{\tau,s}, \quad \Phi_{\tau,\tau} = id.$$

In other words, $\Phi_{\tau,s}$ is the fundamental matrix of the Hamiltonian system associated to the nonstationary quadratic Hamiltonian q_s. We also set $\Gamma_\tau = \{X_\tau v : v \in \mathbb{R}^r\}$, $\Pi_t = T_{\lambda_t}(T^*_{\tilde x(t)}M)$. Then

$$\Lambda_{\tau,t} = \Phi_{\tau,t}(\Pi_t^{\Gamma_\tau}) = \Phi_{\tau,t}(\Pi_t \cap \Gamma_\tau^\angle) + \Gamma_t. \tag{2.4}$$

In particular, all the spaces $\Lambda_{\tau,t}$, $\tau \in [0,t]$ contain the subspace Γ_t; it is also easy to see that the subspaces $\Lambda_{\tau,t}$ contain the line $\mathbb{R}\lambda_t \subset \Pi_t$, this line is in the kernel of the quadratic form q_s, $\forall s$. Hence $\Lambda_{\tau,t} \subset (\Gamma_t + \mathbb{R}\lambda_t)^\angle \subset (T_{\lambda_t}(T^*M))$.

We set

$$\Sigma_{(\xi,\lambda_t)} = (\Gamma_t + \mathbb{R}\lambda_t)^{\angle}/(\Gamma_t + \mathbb{R}\lambda_t); \tag{2.5}$$

then $\Sigma_{(\xi,\lambda_t)}$ is a $2(n-1-r)$-dimensional symplectic space and $\Lambda_{\tau,t}$ are Lagrangian subspaces of $\Sigma_{(\xi,\lambda_t)}$. The curve

$$J_{(\xi,\lambda)} : \tau \mapsto \Lambda_{\tau,t}, \quad \tau \in [0,t],$$

in the Lagrange Grassmannian $L(\Sigma_{(\xi,\lambda_t)}$ will be called the *Jacobi curve* associated with the extremal (ξ,λ).

Originally defined on $[0,t)$ the curve $J_{(\xi,\lambda)}$ has the smooth extension to $[0,t]$ such that $J_{(\xi,\lambda)}(t) = \Pi_t^{(\Gamma_t+\mathbb{R}\lambda_t)}$.

Remark 2.1 *The same construction of the $\mathcal{L}$-derivative and of the Jacobi curve works if one replace conditions (2.1) by their opposite-sign version*

$$\sigma(X_\tau v_1, X_\tau v_2) = 0, \quad \sigma(\dot{X}_\tau v_1, X_\tau v_1) \leq -\alpha|v_1|^2. \tag{2.1$-$}$$

Under conditions (2.1–), the positive inertia index of the $Hess_{(\xi,\lambda_t)}F_{0,t}$ is finite. The difference of these two cases disappears if one defines the Lagrange multiplier λ_t up to a scalar nonzero factor, i.e. as a point in the projectivization of T^*M (see Remark in [4, Sec. 1]).

Suppose that we have an affine in control system:

$$f_\tau(x,u) = g^0(x) + G(x)u,$$

where $G(x) : \mathbb{R}^r \to T_xM$ is a smooth with respect to x family of linear mappings. Then $\frac{\partial f_\tau}{\partial u}(x,u) = G(x)$ and any extremal is singular. The Goh condition has the following form:

$$\lambda_\tau[Gv_1, Gv_2](\tilde{x}(\tau)) = 0, \quad \forall v_1, v_2 \in \mathbb{R}^r, \tau \in [0,t]. \tag{2.6}$$

Besides that, $X_t v = \overrightarrow{\lambda G v}|_{\lambda=\lambda_t}$. If $\tilde{u}(\tau) \equiv 0$, then the (first) generalized Legendre condition takes the form

$$\lambda_\tau[[g^0, Gv], Gv](\tilde{x}(\tau)) \geq 0, \quad \forall v \in \mathbb{R}^r, \tau \in [0,t].$$

To obtain the strenghtened generalized Legendre condition one has to replace 0 by $\alpha|v|^2$ in the right–hand side of the last inequality, for some $\alpha > 0$.

Now consider a linear in control system: $f_\tau(x,u) = G(x)u$. This is, of course, a special case of the affine in control systems, but there is an essential pecularity in the linear case. Namely, if an extremal satisfy the Goh condition, then it may also satisfy the generalized Legendre condition but never the strengthened one. The reason is simple. Any reparametrization of an admissible trajectory of the linear in control system is again an admissible trajectory. At the same time, the reparametrizations do not change the endpoint mapping! This gives a big kernel of the Hessian of the endpoint mapping and implies the degeneration of the generalized Legendre condition. Nevertheless, the described above method for computing the $\mathcal{L}$-derivative works perfectly after a simple modification of the original system.

Let $\tilde u(\tau) \equiv u_0$, $\xi = (u_0, \tilde x)$, and let W be a transversal to u_0 linear hyperplane in $\mathbb{R}^r$, i.e. $\mathbb{R}^r = \mathbb{R}u_0 \oplus W$. Then $((0,\tilde x),\lambda)$ is an extremal of the affine in control system

$$\dot x = G(x)u_0 + G(x)|_W u, \quad u \in W. \tag{2.7}$$

The Goh condition and the (first) generalized Legendre condition for the extremal (ξ,λ) of the original system and the extremal $((0,\tilde x),\lambda)$ of the system (2.7) are equivalent. The strengthened generalized Legendre condition for the given extremal of system (2.7) is equivalent to the inequality

$$\lambda_\tau[[Gu_0, Gv], Gv](\tilde x(\tau)) \geq \alpha |v \wedge u_0|^2, \quad \forall v \in \mathbb{R}^r, \tau \in [0,t], \tag{2.8}$$

for some $\alpha > 0$.

Proposition 2.3 *Let $f_\tau(x,u) = G(x)u$, $\tilde u(\tau) \equiv u_0$. Under conditions (2.6), (2.8), if $\Lambda \subset T_{\lambda_t}(T^*M)$ is the $\mathcal{L}$-derivative of the endpoint mapping of the reduced system (2.7) at $((0,\tilde x),\lambda_t)$, then $\Lambda^{\mathbb{R}\dot\lambda_t}$ is the $\mathcal{L}$-derivative of $F_{0,t}$ at (ξ,λ_t). The same is true if not $\lambda.$ but $(-\lambda.)$ satisfy relations (2.6), (2.8).* □

Let $\Lambda_{\tau,t}$ be the $\mathcal{L}$-derivative of $F_{\tau,t}$ at $(\xi|_{[\tau,t]},\lambda_t)$ and $\Gamma_\tau = \{X_\tau v : v \in \mathbb{R}^r\}$. Then $\Lambda_{\tau,t} \supset (\Gamma_t + \mathbb{R}\lambda_t)$ so that

$$\Lambda_{\tau,t} \subset \Sigma_{(\xi,\lambda_t)} \stackrel{def}{=} (\Gamma_t + \mathbb{R}\lambda_t)^\angle/(\Gamma_t + \mathbb{R}\lambda_t).$$

The curve $J_{(\xi,\lambda)} : \tau \mapsto \Lambda_{\tau,t}$ is the *Jacobi curve* associated with the extremal (ξ,λ). Note that the isotropic subspace Γ_t is defined for the original system, not for the reduced system (2.7) and is, in general, r-dimensional.

Let us consider the case of rank 2 distributions on M, which are linear in control systems with $r = 2$. Any extremal automatically satisfies the Goh condition in this case and there is a nice intrinsic description of the Jacobi curves. So let $\Delta_x \subset T_xM$, $x \in M$, be a rank 2 distribution (a smooth two-dimensional linear subbundle of TM). By Δ^1 we denote the space of all smooth sections of the distribution, $\Delta^1 \subset Vec\, M$. Set

$$\Delta^2 = span\{[g_1, g_2] : g_i \in \Delta^1, i = 1, 2\},$$

$$\Delta^3 = span\{[[g_1, g_2], g_3] : g_i \in \Delta^1, i = 1, 2, 3\},$$

$$\Delta^{i\perp} = \{\lambda \in T^*M : \lambda g(\pi(\lambda)) = 0,\ \forall g \in \Delta^i\}, \quad i = 1, 2, 3,$$

where $\pi : T^*M \to M$ is the canonical projection. It is easy to check that $\Delta^1_x \subset \Delta^2_x \subset \Delta^3_x\ \forall x \in M$, where $\Delta^i_x = \{g(x) : g \in \Delta^i\}$. Hence $\Delta^{3\perp} \subset \Delta^{2\perp} \subset \Delta^{1\perp}$. We also introduce a linear subbundle $\vec{\Delta}_{\lambda_0}$, $\lambda_0 \in T^*M$, in $T(T^*M)$ as follows:

$$\vec{\Delta}_{\lambda_0} = \{\overrightarrow{\lambda g}|_{\lambda=\lambda_0} : g \in \Delta^1\}.$$

Then $T_\lambda(T^*_xM) \subset \vec{\Delta}_\lambda$, $\pi_*\vec{\Delta}_\lambda = \Delta_x$ for any $\lambda \in T^*_xM \setminus \{0\}, x \in M$.

Let $\bar{\sigma}$ be the standard symplectic structure of the symplectic manifold T^*M, then $\sigma = \bar{\sigma}_{\lambda_t}$. To say that (ξ, λ) is an extremal is the same as to say that $\tau \mapsto \lambda_\tau$ is a characteristic curve of $\bar{\sigma}|_{\Delta^{1\perp}}$, i.e $\bar{\sigma}(\dot{\lambda}_\tau, \cdot)|_{\Delta^{1\perp}} = 0$, and that ξ is the 1-jet of $\pi(\lambda.)$. The form $\bar{\sigma}$ is nondegenerate on $(\Delta^{1\perp} \setminus \Delta^{2\perp})$, hence all extremals live in $\Delta^{2\perp}$ and thus satisfy the Goh condition. The extremal $\lambda.$ satisfies the strengthened generalized Legendre condition (or the opposite sign version of this condition) if and only if $\lambda_\tau \in (\Delta^{2\perp} \setminus \Delta^{3\perp})$, $\forall \tau \in [0, t]$.

We assume that $\Delta^2 \neq \Delta^3$; then $(\Delta^{2\perp} \setminus \Delta^{3\perp})$ is a $(2n-3)$-dimensional submanifold in T^*M and

$$\operatorname{rank} \bar{\sigma}_\lambda|_{\Delta^{2\perp}} = (2n-4) \quad \forall \lambda \in (\Delta^{1\perp} \setminus \Delta^{2\perp}).$$

Hence $(\ker \bar{\sigma}_\lambda) \cap T_\lambda\Delta^{2\perp}$ form a line distribution on $(\Delta^{2\perp} \setminus \Delta^{3\perp})$ and characteristics of $\bar{\sigma}_\lambda|_{\Delta^{2\perp}}$ form a smooth *characteristic* 1-foliation of $(\Delta^{1\perp} \setminus \Delta^{2\perp})$.

Suppose that given extremal (ξ, λ) satisfies the strengthened generalized Legendre condition. Then $\tau \mapsto \lambda_\tau$, $\tau \in [0, t]$, is a segment of a leaf of the characteristic foliation. We assume that this segment has no double points. Let $\mathcal{O}_\lambda$ be a tubular neighborhood of $\{\lambda_\tau : 0 \leq \tau \leq t\}$ in $\Delta^{2\perp}$ and let $\mathcal{D}$ be the restriction of the characteristic foliation to $\mathcal{O}_\lambda$ such that the quotient space $\mathcal{O}_\lambda/\mathcal{D}$ is well–defined and is a smooth $2(n-2)$-dimensional manifold.

Let $\nu : \mathcal{O}_\lambda \to \mathcal{O}_\lambda/\mathcal{D}$ be the canonical projection. The form $\bar\sigma_\lambda|_{\Delta^{2\perp}}$ induces a canonical symplecic structure $\hat\sigma$ on the quotient space $\mathcal{O}_\lambda/\mathcal{D}$ by the relation $\nu^*\hat\sigma = \bar\sigma|_{\mathcal{O}}$. Then $T_{\lambda.}(\mathcal{O}_\lambda/\mathcal{D})$ is a $2(n-2)$-dimensional symplectic space. It is easy to see that the line $\nu_*(\mathbb{R}\lambda_\tau) \subset T_{\lambda.}(\mathcal{O}_\lambda/\mathcal{D})$ does not depend on $\tau \in [0,t]$. We set $\mathbb{R}\hat\lambda = \nu_*(\mathbb{R}\lambda_\tau)$ and $\Sigma_\lambda = (\mathbb{R}\hat\lambda)^\angle/\mathbb{R}\hat\lambda$. The symplectic space Σ_λ is naturally isomorphic to the space $\Sigma_{(\xi,\lambda_t)}$ (see (2.5)) for the control problem under consideration. Finally, set

$$J_\lambda(\tau) = \nu_*(\vec\Delta_{\lambda_\tau} \cap T_{\lambda_\tau}\Delta^{2\perp});$$

then $J_\lambda(\tau)$ is a Lagrangian subspace of Σ_λ and the curve $\tau \mapsto J_\lambda(\tau)$ is the Jacobi curve of the extremal (ξ,λ) up to the isomorphism $\Sigma_\lambda \cong \Sigma_{(\xi,\lambda_t)}$.

A similar intrinsic description of Jacobi curves is valid for rank 1 affine distributions on M, i.e. for affine in control systems with scalar controls. Let $\ell_x \subset T_xM \setminus \{0\}$, $x \in M$, be a rank 1 affine distribution (a smooth one-dimensional affine subbundle of TM). We'll consider only normal extremals of this control system, i.e. such that they are not extremals for the rank 2 distribution $span\,\ell_x$, $x \in M$. For the normal extremal (ξ,λ) we have $\lambda_\tau\dot{\hat x}(\tau) \neq 0$. Since $\lambda_\tau\dot{\hat x}(\tau)$ does not depend on τ and $\lambda.$ is defined up to a scalar multiplier, we may (and will) suppose that $\lambda_\tau\dot{\hat x}(\tau) = 1$.

By ℓ^1 we denote the space of all smooth sections of the affine subbundle, $\ell^1 \subset Vec\,M$. Set

$$\ell^0 = \{g_1 - g_2 : g_i \in \ell^1, i = 1,2\},$$
$$\ell^2 = span\{[g_1,g_2] : g_i \in \ell^1, i = 1,2\},$$
$$\ell^3 = span\{[[g_1,g_2],g_3] : g_i \in \ell^1, i = 1,2,3\}.$$

Then ℓ^0 is the space of smooth sections of the one-dimensional vector subbundle associated with given affine subbundle and $\ell^0 \subset \ell^2 \subset \ell^3$. We also introduce dual objects similar to the linear case:

$$\ell^{1\circ} = \{\lambda \in T^*M : \lambda g(\pi(\lambda)) = 1,\ \forall g \in \ell^1\},$$

$$\ell^{i\circ} = \{\lambda \in \ell^{1\circ} : \lambda g(\pi(\lambda)) = 0,\ \forall g \in \ell^i\}, \quad i = 0,2,3,$$
$$\vec{\ell^0_{\lambda_0}} = \{\overrightarrow{\lambda g}|_{\lambda=\lambda_0} : g \in \ell^0\}, \quad \lambda_0 \in T_x^*M \setminus \{0\},\ x \in M.$$

Normal extremals that satisfy the strengthened generalized Legendre condition (or the opposite–sign version of this condition) are just properly parametrized characteristic curves of $\bar\sigma_\lambda|_{(\ell^{2\circ}\setminus\ell^{3\circ})}$. If $\ell^{2\circ} \neq \ell^{3\circ}$, then $\ell^{2\circ} \setminus \ell^{3\circ}$ is a

$(2n-3)$-dimensional submanifold and characteristic curves of $\bar\sigma_\lambda|_{(\ell^{2\circ}\setminus\ell^{3\circ})}$ form a smooth 1-foliation of this submanifold. Let $\tau\mapsto\lambda_\tau$, $\tau\in[0,t]$, be a segment of a leaf of this foliation, $\mathcal{O}_\lambda$ be a tubular neighborhood of $\{\lambda_\tau : 0\le\tau\le t\}$ in $\ell^{2\circ}$, and $\mathcal{D}$ be the restriction of the foliation to $\mathcal{O}_\lambda$ such that the quotient space $\mathcal{O}_\lambda/\mathcal{D}$ is well–defined and is a smooth $2(n-2)$-dimensional manifold. The quotient space $\mathcal{O}_\lambda/\mathcal{D}$ is endowed with a symplectic structure induced by $\bar\sigma_\lambda|_{\ell^{2\circ}}$. The symplectic space $T_{\lambda_\cdot}(\mathcal{O}_\lambda/\mathcal{D})$ is naturally isomorphic to $\Sigma_{(\xi,\lambda_t)}$ (the space $T_{\lambda_\cdot}(\mathcal{O}_\lambda/\mathcal{D})$ does not contain $\mathbb{R}\lambda_\tau$ in contrast to the case of linear distributions). Let $\nu:\mathcal{O}_\lambda\to\mathcal{O}_\lambda/\mathcal{D}$ be the canonical projection. The curve

$$J_\lambda:\tau\mapsto\nu_*(\ell^{\vec{0}}_{\lambda_\tau}\cap T_{\lambda_\tau}\ell^{2\circ})$$

in the Lagrange Grassmannian of $T_{\lambda_\cdot}(\mathcal{O}_\lambda/\mathcal{D})$ is the Jacobi curve of the extremal (ξ,λ) up to the isomorphism $T_{\lambda_\cdot}(\mathcal{O}_\lambda/\mathcal{D})\cong\Sigma_{(\xi,\lambda_t)}$.

We now go back to the general nonlinear system and finish this Section with an expression for the $\mathcal{L}$-derivative which unifies the regular case, the case of Proposition 2.2, and also works in much more degenerate cases. What follows is, in fact, a corollary of results proved in [1].

So we cancel the above assumption on the total singularity of the extremal (ξ,λ) in order to include the regular case. We assume, however, that X_τ is not just Lipschitzian but piecewise smooth with respect to τ; we need this assumption to handle highly degenerate cases. We agree to take all piecewise smooth functions to be left–smooth and to have limits from the right of all derivatives under consideration; the expression $X_\tau^{(k)}v$ below denotes the derivative of $X_\tau v$ of order $k\ge 0$ with respect to τ. Let $m=\dim span\{\pi_*X_\tau v : v\in\mathbb{R}^r, \tau\in(0,t]\}$, then m equals the rank of the differential of the mapping $F_{0,t}$ at ξ. With each $\tau\in(0,t]$ we associate an integer $k_\tau\ge 0$ and a quadratic form b_τ on $\mathbb{R}^r$ as follows: if the form $\frac{\partial^2(\lambda_\theta f_\theta)}{\partial u^2}(\tilde x(\theta),\tilde u(\theta))$ is not identical zero on any interval $\bar\tau<\theta<\tau$, then let $k_\tau=0$ and $b_\tau=\frac{\partial^2(\lambda_\tau f_\tau)}{\partial u^2}(\tilde x(\tau),\tilde u(\tau))$; otherwise, let k_τ be the maximal number k such that $\sigma(X_\theta^{(i)}v_1, X_\theta^{(j)}v_2)\equiv 0$ for $i+j<2(k-1)$ and $v_1,v_2\in\mathbb{R}^r$ on some interval $\bar\tau<\theta<\tau$, and let $b_\tau(v)=\sigma(X_\theta^{(k_\tau)}v, X_\theta^{(k_\tau-1)}v)$, $v\in\mathbb{R}^r$ (if the maximal k exists, then it does not exceed m; if it does not exist, then we set $k_\tau=m+1$ and $b_\tau=0$).

Proposition 2.4 ([1, 2]) *If* $\operatorname{ind}_- Hess_{(\xi,\lambda_t)}F_{0,t}<\infty$, *then*

$$\sigma(X_\tau^{(k_\tau-1)}v_1, X_\tau^{(k_\tau-1)}v_2)=0,\ \forall v_1,v_2\in\mathbb{R}^r,\quad b_\tau\ge 0,\ \tau\in(0,t].$$

If, additionally, $b_\tau(v) \geq \alpha|v|^2$ for some $\alpha > 0$ and all $v \in \mathbb{R}^r, \tau \in (0,t]$, then indeed $\operatorname{ind}_- Hess_{(\xi,\lambda_t)} F_{0,t} < \infty$.

In what follows we assume the sufficient condition in Proposition 2.4 for $\operatorname{ind}_- Hess_{(\xi,\lambda_t)} F_{0,t} < \infty$ to be finite. Let

$$\Gamma_\tau = span\{X_\tau^{(i)} v : 0 \leq i < k_\tau,\ v \in \mathbb{R}^r\},$$

an rk_τ-dimensional isotrpic subspace of $T_{\lambda_t}(T^*M)$.

The quadratic form b_τ is identified with an invertible selfadjoint mapping from $\mathbb{R}^r$ in $\mathbb{R}^{r*}$, and the inverse mapping determines a quadratic form b_τ^{-1} on $\mathbb{R}^{r*}$. The invertibility of b_τ implies that Γ_τ and b_τ are smooth with respect to τ on any interval where X_τ is smooth. In particular, Γ_τ and b_τ^{-1} are piecewise smooth on $[0,t]$.

Set $q_\tau(\eta) = -\frac{1}{2} b_\tau^{-1}(\sigma(X_\tau^{(k_\tau)} \cdot, \eta))$, a nonstationary quadratic Hamiltonian on $T_{\lambda_t}(T^*M)$. Hamiltonian q_τ defines a linear Hamiltonian system on $T_{\lambda_t}(T^*M)$ and, hence, a nonautonomous differential equation on the Lagrange Grassmannian $L(T_{\lambda_t}(T^*M))$. We need the last one, which becomes the matrix Riccati equation in "affine" local coordinates on $L(T_{\lambda_t}(T^*M))$. We prefer coordinate free notations and denote $q_\tau(\Lambda)$ the value at $\Lambda \in L(T_{\lambda_t}(T^*M))$ of the vector field on $L(T_{\lambda_t}(T^*M))$ induced by the Hamiltonian field $\vec{q}_\tau$ on $T_{\lambda_t}(T^*M)$. This notation is coordinated with the canonical identification of the tangent vectors to $L(T_{\lambda_t}(T^*M))$ and quadratic forms (see Append.).

So we have a differential equation

$$\dot{\Lambda} = q_\tau(\Lambda), \quad \Lambda \in L(T_{\lambda_t}(T^*M)); \tag{2.9}$$

what is important and is not quite standard, we allow piecewise smooth (maybe discontinuous in some points) solutions of this equations. Thus to determine a solution of (2.9) we need not only the initial condition but also the relations for "jumps" at the points of discontinuity. In fact, the solution may have a "jump" even at the initial moment since piecewise continuous functions are supposed to be left–smooth but not right–smooth.

Proposition 2.5 *Differential equation (2.9) and the relations*

$$\Lambda(0) = T_{\lambda_t}(T^*_{\tilde{x}(t)}M), \quad \Lambda(\tau+0) = \Lambda(\tau)^{\Gamma_{\tau+0}},\ \tau \in [0,t),$$

imply the inclusions $\Gamma_\tau \subset \Lambda(\tau)$, $\tau \in (0,t]$, and uniquely determine a piecewise smooth curve $\Lambda(\tau), \tau \in [0,t]$, in $L(T_{\lambda_t}(T^*M))$. Then $\Lambda(t)$ is the $\mathcal{L}$-derivative of $F_{0,t}$ at (ξ, λ_t). $\square$

Theorem 2.1 *Under conditions of Proposition 2.5, let $\tau_1, \ldots, \tau_l$ be all points of discontinuity of the curve $\Lambda(\cdot)$. Suppose $\bar{\Lambda}(\cdot)$ is a continuous closed curve in the Lagrange Grassmannian obtained from the curve $\tau \mapsto \Lambda(t-\tau)$ by the connecting of $\Lambda(\tau_i+0)$ with $\Lambda(\tau_i)$, $i = 1, \ldots, l$, and $\Lambda(0)$ with $\Lambda(t)$ by simple nondecreasing curves (see Append.). Then*

$$\operatorname{ind}_- Hess_{(\xi,\lambda_t)} F_{0,t} = \operatorname{ind} \bar{\Lambda}(\cdot) - m,$$

where $\operatorname{ind} \bar{\Lambda}(\cdot)$ *is the Maslov index of* $\bar{\Lambda}(\cdot)$. $\square$

3 Three-dimensional systems

We are going to study generic germs of affine in control systems admitting extremals, and we are going to do it in a lowest possible dimension. Generic germs do not admit extremals if $n = 2$ and if $n = 3, r = 2$, therefore we focus on the case $n = 3, r = 1$. So we deal with the germ at $x_0 \in M$ of a system

$$\dot{x} = g^0(x) + ug^1(x), \ u \in \mathbb{R}, \ x \in M, \quad \dim M = 3. \tag{3.1}$$

Generic assumptions are as follows:

$$\begin{gathered} g^0(x) \wedge g^1(x) \wedge [g^0, g^1](x) \neq 0, \\ g^1(x) \wedge [g^1, g^0](x) \wedge [g^1, [g^1, g^0]](x) \neq 0. \end{gathered} \tag{3.2}$$

We'll describe a normal form of the germ and a complete system of invariants under feedback and state transformations.

First let us roughly evaluate the "number of parameters". Affine lines in $\mathbb{R}^3$ form a 4-dimensional manifold, hence the space of the equivalence classes of systems of form (3.1) with respect to purely feedback (not the state!) transformations can be locally parametrized by 4 germs of real smooth functions of 3 variables. The group of state transformations is parametrized by 3 smooth functions of 3 variables. We thus may expect $4-3 = 1$ arbitrary germ of a smooth function of 3 variables as a principal invariant and a number of germs of functions of 2, 1, and 0 variables as some additional invariants.

We have $\ell_x = g^0(x) + \mathbb{R}g^1(x)$,

$$\ell^{2\circ} = \{\lambda \in T^*M : \lambda g^0 = 1, \ \lambda g^1 = \lambda[g^0, g^1] = 0\}, \quad \ell^{3\circ} = \emptyset.$$

Hence $\ell^{2\circ} = \ell^{2\circ} \setminus \ell^{3\circ}$ is just a section of the bundle T^*M and $\ell^{2\circ}$ is foliated by the extremals. In other words, extremal trajectories form a local flow in

M; there exists a unique smooth function v on M such that the extremal trajectories are exactly trajectories of the differential equation $\dot{x} = g^0(x) + v(x)g^1(x)$. Replacing g^0 by $g^0 + vg^1$, if necessary, we may assume from the very beginning that the trajectories of the system $\dot{x} = g^0(x)$ are the extremal trajectories. The last property is equivalent to the identity

$$g^0(x) \wedge [g^0, g^1](x) \wedge [g^0, [g^0, g^1]](x) = 0, \quad x \in M. \tag{3.3}$$

The only feedback transformations preserving identity (3.3) are the multiplications of g^1 by a nonvanishing function.

The extremals do not admit reparametrizations and a Jacobi curve is associated to every extremal. The Jacobi curves are curves in the Lagrange Grassmannian of the symplectic space of dimension $2(3-2) = 2$. In other words, these are curves in the projective line. The curvature tensors associated to germs of regular curves in the Lagrange Grassmannians are just real numbers in this low dimensional case (see [4, Sec. 4]). Through every $x \in M$ passes exactly one extremal trajectory corresponding to a uniquely determined extremal in $\ell^{2\circ}$. Take the germ of the extremal with the endpoint $\lambda_t = \ell^{2\circ} \cap T_x M$. Assume that $R(x)$ is the curvature of the germ at t of the Jacobi curve associated to the extremal. We'll see that $R(x)$, $x \in M$, is a smooth function of x and the function R turns into the required principal functional invariant as soon as one fix some canonical coordinates in the neighborhood of x_0.

Let us denote $g^2 = [g^0, g^1]$; then $g^0(x), g^1(x), g^2(x)$ form a basis of $T_x^* M$ and we have

$$[g^0, g^1] = g^2, \quad [g^2, g^0] = c_{02}^1 g^1 + c_{02}^2 g^2, \quad [g^2, g^1] = \sum_{k=0}^{2} c_{12}^k g^k,$$

where the "structural constants" c_{ij}^k are actually smooth functions on M and $c_{12}^0 \neq 0$.

Proposition 3.1 $R = c_{02}^1 - \frac{1}{4}(c_{02}^2)^2 - \frac{1}{2} g^0 c_{02}^2$.

Proof. Let $\omega_0, \omega_1, \omega_2$ be the frame in T^*M dual to the frame g^0, g^1, g^2 in TM, i.e. $\langle \omega_i, g^j \rangle = \delta_{ij}$, $i, j = 0, 1, 2$. Then $\ell^{2\circ} \subset T^*M$ is the graph of ω_0,

$$\bar{\sigma}|_{\ell^{2\circ}} = (\pi|_{\ell^{2\circ}})^* \left(c_{12}^0 \omega_1 \wedge \omega_2\right), \quad \vec{\ell^0} \cap T\ell^{2\circ} = (\pi|_{\ell^{2\circ}})_*^{-1} g^1.$$

The diffeomorphism $\pi|_{\ell^{2\circ}}$ provides the identification of $\ell^{2\circ}$ with M and we make the remaining calculations in M instead of $\ell^{2\circ}$. Let $\tau \mapsto J_x(\tau)$ be the

Jacobi curve associated to the extremal trajectory $\tau \mapsto xe^{(\tau-t)g^0}$. Then $J_x(\tau)$ is a line in the plane $T_xM/(\mathbb{R}g^0(x))$ endowed with the symplectic structure $(c^0_{12}\omega_1 \wedge \omega_2)_x$. More precisely, $J_x(\tau) = \mathbb{R}\left(e_*^{(t-\tau)g^0} g^1\right)(x) + \mathbb{R}g^0(x)$. The vectors $g^1(x)$, $\frac{1}{c^0_{12}(x)} g^2(x)$ form a canonical basis of the symplectic plane; what we need are the coordinates of the vector $\left(e_*^{(t-\tau)g^0} g^1\right)(x) = \left(e^{(\tau-t)ad\, g^0} g^1\right)(x)$ in this basis. We have $e^{(\tau-t)ad\, g^0} g^1 = a^1_\tau g^1 + a^2_\tau g^2$, where $a^1_t = \dot{a}^2_t = 1$, $\dot{a}^1_t = a^2_t = 0$,

$$\ddot{a}^i_\tau = -\left(e^{(\tau-t)g^0} c^1_{02}\right) a^i_\tau - \left(e^{(\tau-t)g^0} c^2_{02}\right) \dot{a}^i_\tau, \quad i = 1, 2; \tag{3.4}$$

here $\dot{}$ is the differentiation with respect to τ.

According to [4, Sec. 4], the curvature $R(x)$ is the Schwartzian derivative at t of the function $S_\tau = c^0_{12}(x)\frac{a^2_\tau(x)}{a^1_\tau(x)}$,

$$R(x) = \frac{1}{2}\frac{\dddot{S}_t}{\dot{S}_t} - \frac{3}{4}\left(\frac{\ddot{S}_t}{\dot{S}_t}\right)^2. \tag{3.5}$$

The desired expression for R is derived from (3.4), (3.5) by a straightforward calculation. $\square$

The curvature R is feedback–invariant, while the "structural constants" c^k_{ij} are not. A feedback transformation $g^1 \mapsto bg^1$, where $b \in C^\infty(M)$, transforms $c^0_{12} \mapsto b^2 c^0_{12}$. Hence $\operatorname{sign} c^0_{12}$ is feedback–invariant and the vector field $\hat{g} = |c^0_{12}|^{-\frac{1}{2}} g^1$ may change only the sign under feedback transformations.

We put $\nu = \operatorname{sign} c^0_{12}$ and define a feedback invariant germ N of a two-dimensional submanifold of M as follows:

$$N = \{x_0 e^{y_2[g^0,\hat{g}]} \circ e^{y_1\hat{g}} : y_1, y_2 \text{ are close to 0 real numbers}\}.$$

Lemma 3.1 *There is a unique up to a sign feedback normalization of the germ of g^1 at x_0 such that the following identities hold for the normalized germ:*

$$c^2_{02} \equiv 0, \quad |c^0_{12}|\Big|_N \equiv 1 \tag{3.6}$$

Proof. We set $g^1|_N = \hat{g}|_N$. Note that the field g^1 is tangent to N. The field g^0 is transversal to N, therefore the trajectories of g^0 define a tubular neighborhood of N in M. A renormalization $g^1 \mapsto bg^1$ transforms $c^2_{02} \mapsto$

$c_{02}^2 - \frac{2}{b} g^0 b$. The renormalized field satisfies the required conditions if and only if $b|_N = \pm 1$, $g^0 b = \frac{1}{2} b c_{02}^2$. The second identity is equivalent to the following one:

$$e^{tg^0} b = b e^{\frac{1}{2}\int_0^t e^{\tau g^0} c_{02}^2 \, d\tau}.$$

This means that b is uniquely defined on a trajectory of the field g^0 as soon as we fix b at a point of the trajectory, and this is all we need to complete the proof. $\square$

Assume that g^1 is normalized and (3.6) is valid. We introduce *normal coordinates* (y_0, y_1, y_2) in a neighborhood of x_0 by the formula

$$\Phi(y_0, y_1, y_2) = x_0 e^{y_2 g^2} \circ e^{y_1 g^1} \circ e^{y_0 g^0}.$$

In particular, $\Phi^{-1}(N)$ is the coordinate plane $y_0 = 0$.

Let $t \mapsto \rho_i(t, y_1, y_2)$, $i = 0, 1$, be the solutions of the Hill equation $\ddot{\rho} + R(t, y_1, y_2)\rho = 0$ with the initial conditions

$$\begin{pmatrix} \rho_0(0, y_1, y_2) & \dot{\rho}_0(0, y_1, y_2) \\ \rho_1(0, y_1, y_2) & \dot{\rho}_1(0, y_1, y_2) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

In the coordinates $y = (y_0, y_1, y_2)$, the fields g^0, g^1 have the form:

$$g^0 = \frac{\partial}{\partial y_0}, \quad g^1 = \rho_0(y) \frac{\partial}{\partial y_1} + \rho_1(y) \sum_{i=0}^{2} \alpha_i(y_1, y_2) \frac{\partial}{\partial y_i}, \tag{3.7}$$

where α_1, α_2 are arbitrary germs of smooth functions meeting the conditions

$$\alpha_1(0, y_2) = 0, \quad \alpha_2(0, y_2) = 1, \tag{3.8}$$

and $\alpha_0(y_1, y_2) = -\nu \int_0^{y_1} \frac{\alpha_2(y_1, y_2)}{\alpha_2(t, y_2)} \, dt$. The only ambiguity of the construction is the sign of g^1. Changing the sign leads to the transformation

$$\begin{gathered} R(y_1, y_2, y_3) \mapsto R(y_1, -y_2, -y_3), \\ (\alpha_1(y_1, y_2), \alpha_2(y_1, y_2)) \mapsto (\alpha_1(-y_1, -y_2), \alpha_2(-y_1, -y_2)). \end{gathered} \tag{3.9}$$

Summing up, we obtain the following

Proposition 3.2 *Any system of the form (3.1) subject to conditions (3.2) has a local normal form (3.7) under smooth feedback and state transformations. In the normal form, R is an arbitrary germ of smooth function on $\mathbb{R}^3$, α_1, α_2 are arbitrary germs of smooth function on $\mathbb{R}^2$ meeting conditions (3.8); ρ_0, ρ_1 are determined by R, α_0 is determined by α_2 and the discrete invariant $\nu \in \{-1, 1\}$. The germs R, α_1, α_2 are feedback and state invariants up to the involution (3.9).* $\square$

Recall that R is the curvature, i.e. a well–defined function on M with a clear geometric and variational meaning. The invariant ν has the following nice interpretation. Small enough pieces of the extremal trajectories realize local extrema in the time optimal problem for system (3.1); they are local time minimizers if $\nu = -1$ and local time maximizers if $\nu = 1$. Invariants α_1, α_2 are well–defined germs of smooth functions on N; the geometric meaning of these germs is not so clear yet.

In the "flat" case $R \equiv 0, \alpha_1 \equiv 0, \alpha_2 \equiv 1$ we obtain the relations

$$[g^0, [g^0, g^1]] = 0, \quad [g^1, [g^1, g^0]] = \nu g^0.$$

The fields g^0, g^1 thus generate a solvable three-dimensional Lie algebra: the Lie algebra of the group of isometries of the Euclidean plane for $\nu = -1$ and of the pseudo-Euclidean plane for $\nu = 1$. Then system (3.1) is locally equivalent to a left–invariant system on the corresponding group of isometries.

Remark 3.1 *Actually, the results of this section give more than a normal form in a neighborhood of a point. It is easy to see that the same normal form is valid in a neighborhood of an as long as we want segment of the extremal trajectory, while the trajectory has no double points and conditions (3.2) are satisfied.*

4 Appendix. On Lagrange Grassmannians

Here we introduce some notions of linear symplectic geometry that we use in the paper (see [3, 7, 10, 11] for more details). A *symplectic structure* in an even-dimensional linear space Σ is defined by a nondegenerate bilinear skew-symmetric 2-form $\sigma(\cdot, \cdot)$. Two vectors $\xi_1, \xi_2 \in \Sigma$ are called skew-orthogonal, written $\xi_1 \angle \xi_2$, if $\sigma(\xi_1, \xi_2) = 0$. If N is a subspace of Σ, let us denote by $N^\angle$ its skew-orthogonal complement: $N^\angle = \{\xi \in \Sigma \mid \sigma(\xi, \nu) = 0, \forall \nu \in N\}$.

Evidently $\dim N + \dim N^{\angle} = \dim \Sigma$. A subspace $\Gamma \subseteq \Sigma$ is called *isotropic*, if $\Gamma \subseteq \Gamma^{\angle}$, and *coisotropic*, if $\Gamma \supseteq \Gamma^{\angle}$. A subspace $\Lambda \subset \Sigma$ is called *Lagrangian*, if $\Lambda^{\angle} = \Lambda$. Such subspaces have dimension $\frac{1}{2}\dim \Sigma$. If Λ is a Lagrangian subspace and Γ is isotropic, then it is easy to prove, that $(\Lambda \cap \Gamma^{\angle}) + \Gamma = (\Lambda + \Gamma) \cap \Gamma^{\angle}$ is a Lagrangian subspace. We denote it by Λ^{Γ}.

The symplectic group $Sp(\Sigma)$ is the group of those linear transformations of Σ, which preserve the symplectic form:

$$Sp(\Sigma) = \{S \in GL(\Sigma) : \sigma(S\xi_1, S\xi_2) = \sigma(\xi_1, \xi_2)\ \forall \xi_1, \xi_2 \in \Sigma\}.$$

The elements of this group are called symplectic transformations of Σ. The Lie algebra of the symplectic group is:

$$sp(\Sigma) = \{A \in gl(\Sigma) : \sigma(A\xi_1, \xi_2) = \sigma(A\xi_2, \xi_1)\ \forall \xi_1, \xi_2 \in \Sigma\}.$$

Let H be a real quadratic form on Σ and $d_\xi H$ be the differential of H at a point $\xi \in \Sigma$. Then $d_\xi H$ is a linear form on Σ which depends linearly on ξ. For every $\xi \in \Sigma$ there exists a unique vector $\vec{H}(\xi) \in \Sigma$ which satisfies equality $\sigma(\cdot, \vec{H}(\xi)) = d_\xi H$. It is easy to show that the linear operator $\vec{H}\colon \Sigma \to \Sigma$ belongs to $sp(\Sigma)$, and the mapping $H \mapsto \vec{H}$ is an isomorphism of the space of quadratic forms onto $sp(\Sigma)$. The differential equation $\dot{\xi} = \vec{H}(\xi)$ is called the linear Hamiltonian system corresponding to the quadratic Hamiltonian H.

Denote by $L(\Sigma)$ the Grassmanian of Lagrangian subspaces of Σ. This is a smooth compact manifold of dimension $\frac{1}{8}\dim \Sigma(\dim \Sigma + 2)$.

Certainly symplectic transformations transform Lagrangian subspaces into Lagrangian ones, hence the symplectic group acts on $L(\Sigma)$. It is easy to show that it acts transitively.

Let us consider a tangent space $T_\Lambda L(\Sigma)$, $\Lambda \in L(\Sigma)$. To every quadratic form h on Σ there corresponds a linear Hamiltonian vector field $\vec{h}$ and a one-parameter subgroup $t \mapsto e^{t\vec{h}}$ in $Sp(\Sigma)$. Let us consider the linear mapping $2h \mapsto \frac{d}{dt} e^{t\vec{h}}(\Lambda)|_{t=0}$ of the space of quadratic forms into $T_\Lambda L(\Sigma)$. This mapping is surjective and its kernel consists of all quadratic forms which vanish on Λ. Thus two different quadratic forms correspond to the same vector from $T_\Lambda L(\Sigma)$ if and only if the restrictions of these forms on Λ coincide. Hence we obtain a natural identification of the space $T_\Lambda L(\Sigma)$ with the space of quadratic forms on Λ.

A tangent vector $\eta \in T_\Lambda L(\Sigma)$ is called nonnegative if the corresponding quadratic form is nonnegative on Λ. An absolutely continuous curve Λ_τ ($\tau \in$

$[0,t]$) in $L(\Sigma)$ is called nondecreasing if the velocities $\dot{\Lambda}_\tau \in T_{\Lambda_\tau}L(\Sigma)$ are nonnegative for almost all $\tau \in [0,T]$.

Treating the action of symplectic group $Sp(\Sigma)$ on $L(\Sigma)$ one can easily verify, that pairs of Lagrangian subspaces (Λ, Λ') have only one invariant w.r.t. this action: it is $\dim(\Lambda \cap \Lambda')$. For triples of Lagrangian subspaces, there are more invariants.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be Lagrangian subspaces. Let us present a vector $\lambda \in (\Lambda_1 + \Lambda_3) \cap \Lambda_2$ as a sum $\lambda = \lambda_1 + \lambda_3$ and consider a properly defined on $(\Lambda_1+\Lambda_3)\cap\Lambda_2$ quadratic form $S(\lambda) = \sigma(\lambda_1, \lambda_3)$. *Maslov index* $\mu(\Lambda_1, \Lambda_2, \Lambda_3)$ of the triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ is the signature of $S(\lambda)$. It is an invariant of the action of symplectic group. Maslov index is anti-symmetric and satisfies the chain rule: $\mu(\Lambda_1, \Lambda_2, \Lambda_3) = -\mu(\Lambda_2, \Lambda_1, \Lambda_3) = -\mu(\Lambda_1, \Lambda_3, \Lambda_2)$, $\mu(\Lambda_1, \Lambda_2, \Lambda_3) - \mu(\Lambda_1, \Lambda_2, \Lambda_4) + \mu(\Lambda_1, \Lambda_3, \Lambda_4) - \mu(\Lambda_2, \Lambda_3, \Lambda_4) = 0$.

In [1] a bit different invariant of a triple of Lagrangian planes $(\Lambda_1, \Lambda_2, \Lambda_3)$ was exploited for computation of Morse index for singular extremals.

Definition 4.1 *Consider the quadratic form* $S(\lambda) = \sigma(\lambda_1, \lambda_3)$ *with the domain* $((\Lambda_1 + \Lambda_3) \cap \Lambda_2)/\bigcap_{i=1}^3 \Lambda_i$. *A sum* $\frac{1}{2}\dim\ker S + \mathrm{ind}_- S$, *where* $\mathrm{ind}_- S$ *is the negative inertia index of* S, *is an invariant of the triple* $(\Lambda_1, \Lambda_2, \Lambda_3)$ *of Lagrangian subspaces. It is denoted by* $\mathrm{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3)$ *and is called Maslov-type index.*

Let us note, that $\ker S = ((\Lambda_1 \cap \Lambda_2) + (\Lambda_2 \cap \Lambda_3))/\bigcap_{i=1}^3 \Lambda_i$. We refer to [1] for a simple formula connecting this Maslov-type index with Maslov index of the triple and for the proof of the following 'triangle inequality':

$$\mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_3) \le \mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) + \mathrm{ind}_{\Lambda_0}(\Lambda_2, \Lambda_3).$$

It also follows directly from the definition, that:

$$\mathrm{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3) = \dim \Lambda_1 \quad \Leftrightarrow \quad \mu(\Lambda_1, \Lambda_2, \Lambda_3) = -\dim \Lambda_1; \tag{4.1}$$

$$\mathrm{ind}_{\Lambda_1}(\Lambda_1, \Lambda_3) = \frac{1}{2}\dim\ker S = \frac{1}{2}(\dim \Lambda_1 - \dim(\Lambda_1 \cap \Lambda_3)).$$

A continuous curve $\Lambda(\tau) \in L(\Sigma)$, $0 \le \tau \le 1$, is called *simple* if there exists $\Delta \in \mathcal{L}(\Sigma)$ such that $\Lambda(\tau) \cap \Delta = 0 \ \forall \tau \in [0,1]$.

Lemma 4.1 *If* $\Lambda(\tau) \in L(\Sigma)$, $0 \le \tau \le 1$, *is a simple nondecreasing curve in* $\mathcal{L}(\Sigma)$, *and* $\Pi \in \mathcal{L}(\Sigma)$, *then*

$$\mathrm{ind}_\Pi(\Lambda(0), \Lambda(1)) = \mathrm{ind}_\Pi(\Lambda(0), \Lambda(\tau)) + \mathrm{ind}_\Pi(\Lambda(\tau), \Lambda(1)), \ \forall \tau \in [0,1].$$

Lemma 4.2 *Let $\Lambda^0, \Lambda^1 \in L(\Sigma)$. There exist $\Delta \in L(\Sigma)$ and neighborhoods $V^0 \ni \Lambda^0$, $V^1 \ni \Lambda^1$ in $\mathcal{L}(\Sigma)$ such that whenever $\Lambda \in V^0, \Lambda' \in V^1$ and $\dim(\Lambda \cap \Lambda') = \dim(\Lambda^0 \cap \Lambda^1)$ then there exists a simple nondecreasing curve $\Lambda(\tau),\ \tau \in [0,1]$ such that $\Lambda(0) = \Lambda, \Lambda(1) = \Lambda',\ \Lambda(\tau) \cap \Delta = 0\ \forall \tau \in [0,1]$.*

Both Lemmas are proved in [1].

Definition 4.2 *Let $\Lambda(\tau),\ 0 \leq \tau \leq t$, be a nondecreasing curve in $L(\Sigma)$ and $0 = \tau_0 < \tau_1 < \cdots < \tau_l = t$ are such, that the curves $\Lambda(\cdot)|_{[\tau_i, \tau_{i+1}]}$, $i = 0, \ldots l-1$, are simple and $\Pi \in L(\Sigma)$. The expression*

$$\mathrm{ind}_\Pi \Lambda(\cdot) = \sum_{i=0}^{l-1} \mathrm{ind}_\Pi(\Lambda(\tau_i), \Lambda(\tau_{i+1})) \tag{4.2}$$

is called Maslov index of the curve $\Lambda(\cdot)$ with respect to Π.

It follows from Lemma 4.1 that (4.2) does not depend on a choice of $\tau_1 < \cdots < \tau_{l-1}$. If $\Lambda(0) \cap \Pi = \Lambda(t) \cap \Pi = 0$, then the intersection number $\Lambda(\cdot) \cdot \mathcal{M}_\Pi$ of the curve $\Lambda(\cdot)$ with the hypersurface (train) $\mathcal{M}_\Pi = \{\Delta \in L(\Sigma) : \Delta \cap \Pi \neq 0\}$ is well–defined and the identity $\Lambda(\cdot) \cdot \mathcal{M}_\Pi = \mathrm{ind}_\Pi \Lambda(\cdot)$ is valid. If the curve $\Lambda(\tau)$ is closed ($\Lambda(0) = \Lambda(t)$), then $\mathrm{ind}_\Pi \Lambda(\cdot)$ does not depend also on the choice of Π and is denoted $\mathrm{ind}\,\Lambda(\cdot)$ (cf. [1]).

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