On Morse Theory for Piecewise Smooth Functions

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Abstract

Lower level sets of continuous selections of C^2-functions defined on a smooth manifold in the vicinity of a nondegenerate critical point in the sense of [11] are studied. It is shown that the lower level set is homotopy equivalent to the join of the lower level sets of the smooth and the nonsmooth part, respectively, of the corresponding normal form. Some generalized Morse inequalities are deduced from this result.

Key Words: Continuous selection, critical point, nonsmooth Morse theory

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1 Introduction

For a smooth function $f : M \longrightarrow \mathbb{R}$ defined on a smooth n-dimensional manifold M, Morse Theory studies the topological types of the lower level sets

$$M^\alpha = \{ \ x \mid f(x) \leq \alpha\}$$

for $\alpha \in \mathbb{R}$ (cf. [13]). The first Morse Lemma states that for a C^1-function f and $\alpha < \beta$ the lower level set M^α is a strong deformation retract of M^β provided that M^β is compact and $f^{-1}([\alpha, \beta])$ does not contain a critical point. The second Morse Lemma is concerned with the local behaviour of C^2-functions in a neighborhood of a nondegenerated critical point $x_0 \in M$. It states that whenever the Hessian of f at x_0 is regular, the function f is locally topologically equivalent in local coordinates to a function

$$q(y) = f(x_0) - \sum_{j=1}^{l} y_j^2 + \sum_{j=l+1}^{n} y_j^2$$

where $l \in \{0, \ldots, n\}$ is the index of the critical point $x_0 \in M$. A version of the second Morse Lemma for piecewise smooth functions can be found in [11]. The latter result is the starting point for our study.

A version of Morse theory for smooth functions restricted to nonsmooth subvarieties of a smooth manifold is developed in [7] (see also [1]). In contrast to this work we shall be concerned with extentions of Morse theory to nonsmooth functions defined on smooth manifolds. Some preliminary work in this direction has been done in [4, 5]. Following [11], we shall use a critical point concept for locally Lipschitz continuous functions based on Clarke's generalized derivative [6] which is defined by

$$f^o(x_0, y) = \limsup_{\substack{x \to x_0 \\ \alpha \to 0 \\ \alpha > 0}} \frac{f(x + \alpha y) - f(x)}{\alpha}.$$

The function $f^o(x_0, .)$ is sublinear; its support set is called Clarke's subdifferential of f at x_0 and denoted by $\partial f(x_0)$; i.e.

$$f^o(x_0, y) = \max_{v \in \partial f(x_0)} \langle v, y \rangle.$$

The important chain rule [5] implies that the Clarke subdifferential of a locally Lipschitz continuous function $f : M \longrightarrow \mathbb{R}$ defined on a smooth manifold M has an invariant meaning, i.e. does not depend on the choice of local coordinates in M. More precisely, $\partial f(x)$ is a compact convex subset in $T_x^* M$.

Definition 1.1 *Let M be a smooth n-dimensional manifold, $f : M \longrightarrow I\!R$ be a locally Lipschitz-continuous function. A point $x_0 \in M$ is called a critical point of f if $0 \in \partial f(x_0)$.*

The following proposition extends the first Morse Lemma to locally Lipschitzian functions.

Proposition 1.2 *Let $f : M \to R$ be a locally Lipschitz continuous function, $M_t = f^{-1}((-\infty, t])$. Suppose M_β is compact and $f^{-1}([\alpha, \beta])$ does not contain critical points of f. Then there exists a Lipschitz continuous mapping $F : M_\beta \times [0, 1] \to M_\beta$ such that*

$$F(M_\beta, t) \subset M_{\beta + t(\alpha - \beta)}, \quad F \mid_{M_\alpha \times [0,1]} = id.$$

Proof. Take $x \in f^{-1}([\alpha, \beta])$. Since x is not a critical point for f, there exists $v(x) \in T_x M$ such that $\sup\limits_{\xi \in \partial f(x)} \langle \xi, v(x) \rangle < 0$. Choose a vector field v on M with value $v(x)$ at x. The upper semi-continuity of the Clarke subdifferential implies that $\sup\limits_{\xi \in \partial f(y)} \langle \xi, v(y) \rangle < -\varepsilon_x$ for any $y \in O_x$, where O_x is a neighborhood of x in M and $\varepsilon_x > 0$. The neighborhoods O_x, $x \in f^{-1}([\alpha, \beta])$, form a covering of $f^{-1}([\alpha, \beta])$, let O_{x_i}, $i = 1, \ldots, m$, be a finite subcovering and let v_i be the corresponding smooth vector fields on M, i.e. $\sup\limits_{\xi \in \partial f(y)} \langle \xi, v_i(y) \rangle < -\varepsilon$, $\forall y \in O_{x_i}$, where $\varepsilon = \min\limits_{1 \le i \le m} \varepsilon_{x_i}$.
Suppose φ_i, $i = 1, \ldots, m$, is a smooth partition of unity subordinated to the covering O_{x_i}, $i = 1, \ldots, m$, of $f^{-1}([\alpha, \beta])$. This means that $\varphi_i(x) \ge 0 \; \forall x \in M$, $\varphi_i(x) = 0 \; \forall x \in M \setminus$

O_{x_i}, and $\sum\limits_{i=1}^{m} \varphi_i(x) = 1 \ \forall x \in f^{-1}([\alpha,\beta])$. We set $v(x) = \sum\limits_{i=1}^{m} \varphi_i(x) v_i(x)$. Then v is a smooth vector field on M; $\sup\limits_{\xi \in \partial f(x)} \langle \xi, v(x) \rangle < -\varepsilon \ \forall x \in f^{-1}([\alpha,\beta])$; $v(x) = 0 \ \forall x \in M \setminus \bigcup\limits_{i=1}^{m} O_{x_i}$. We may assume without lack of generality that $\bigcup\limits_{i=1}^{m} O_{x_i}$ has a compact closure. Then the vector field v has a compact support and is thus a complete field.
Let us denote by $t \mapsto y(t;x)$, $x \in f^{-1}([\alpha,\beta])$, the solution of the differential equation $\dot{y} = v(y)$ with the initial condition $y(0;x) = x$. Then $t \mapsto f(y(t;x))$ is a scalar Lipschitz continuous function and the derivative of this function, which is defined almost everywhere, satisfies the inequalities $\frac{d}{dt} f(y(t;x)) \leq 0 \ \forall t$ and $\frac{d}{dt} f(y(t;x)) < -\varepsilon$, if $f(y(t;x)) \in [\alpha,\beta]$. Hence there exists the Lipschitz continuous function $\tau \mapsto a(\tau;x)$, $\tau \in [\alpha, f(x)]$, which is inverse to the function $t \mapsto f(y(t;x))$. Moreover, it follows from the Lipschitzian version of the implicit function theorem [6] that a is Lipschitz continuous as a function of the pair $(\tau;x)$, where $x \in f^{-1}([\alpha,\beta])$, $\tau \in [\alpha, f(x)]$. We have $f(y(a(\tau;x);x)) = \tau$; thus

$$F(x,t) = \begin{cases} y(a((\alpha - f(x))t + f(x);x);x), & \text{if } \alpha \leq f(x) \leq \beta; \\ x, & \text{if } f(x) \leq \alpha \end{cases}$$

satisfies the assertion. □

2 The second Morse Lemma for piecewise smooth functions

Definition 2.1 *Let M be an n-dimensional topological manifold, and let $f : M \longrightarrow \mathbb{R}$ and $f_1, \ldots, f_m : M \longrightarrow \mathbb{R}$ be continuous functions. If $I(x) = \{i \in \{1,...,m\} | f_i(x) = f(x)\}$ is nonempty at every point $x \in M$, then f is called a continuous selection of the functions $f_1, \ldots, f_m$. We denote by $CS(f_1, \ldots, f_m)$ the set of all continuous selections of $f_1, \ldots, f_m$. The set $I(x)$ is called the active index set of f at the point x.*

Typical examples of continuous selection of the functions $f_1, \ldots, f_m$ are functions of the form

$$f(x) = \max_{i \in I} \min_{j \in J_i} f_j(x),$$

where I is a finite index set and $J_i \subseteq \{1, \ldots, m\}$. We shall confine ourselves to continuous selections of smooth functions. A continuous selection f of C^1-functions $f_1, \ldots, f_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ is locally Lipschitz continuous and Clarke's subdifferential of f is given by

$$\partial f(x) = \mathrm{conv}\{\nabla f_i(x) | i \in \hat{I}(x)\},$$

where

$$\hat{I}(x) = \{i | x \in \mathrm{clint}\{x | f(x) = f_i(x)\}\},$$

and "conv", "cl", and "int" denote the convex hull, the closure, and the interior operators on the respective sets, [14]. In the sequel we use the following notion of a nondegeneracy critical point for continuous selections of C^2-functions (cf. [11]).

Definition 2.2 *Let M be a smooth n-dimensional manifold, $f_1, \ldots, f_m : M \longrightarrow I\!R$ C^2-functions, and $f \in CS(f_1, \ldots, f_m)$. A critical point $x_0 \in M$ of f is called nondegenerate if the following two conditions hold:*

(ND1) For each $i \in \hat{I}(x_0)$, the set of differentials $\{df_j(x_0) | j \in \hat{I}(x_0) \backslash \{i\}\}$ is linearly independent;

(ND2) The second differential $d^2L(., \hat{\lambda})(x_0)$ of $x \mapsto L(x, \hat{\lambda})$ is regular on

$$\hat{T}(x_0) = \bigcap_{i \in \hat{I}(x_0)} kern\ df_i(x_0),$$

where

$$L(x, \lambda) = \sum_{i \in \hat{I}(x_0)} \lambda_i f_i(x),$$

and the reals $\hat{\lambda}_i$ are such that

$$dL(., \hat{\lambda})(x_0) = 0, \quad \sum_{i \in \hat{I}(x_0)} \hat{\lambda}_i = 1, \ \hat{\lambda}_i \geq 0 \text{ for every } i \in \hat{I}(x_0).$$

The dimension of a maximal linear subspace of $\hat{T}(x_0)$ on which $d^2L(., \hat{\lambda})(x_0)$ is negative definite is called the quadratic index of the critical point x_0.

Note that the reals $\hat{\lambda}_i$, $i \in \hat{I}(x_0)$, in (ND2) exist and are unique in the presence of (ND1). If f is twice continuously differentiable at a critical point, then (ND2) corresponds to the classical nondegeneracy condition. Geometrically speaking, a point x_0 is a nondegenerate critical point of the continuous selection f if the graphs of the functions f_j, $j \in \hat{I}(x_0)$, have a transversal intersection at the point $(x_0, f(x_0))$ and x_0 is a nondegenerate critical point of the restriction of the mapping $(x, f(x)) \mapsto f(x)$ to the intersection of these graphs. Note that the intersection of the graphs is a smooth submanifold and that $(x, f(x))$ is a smooth function on this submanifold.

The following result of [11] is the basis for our study (cf. also [12] for an alternative proof).

Theorem 2.3 *Let $U \subseteq I\!R^n$ be an open subset, $x_0 \in U$, $f_1, \ldots, f_m : U \longrightarrow I\!R$ be twice continuously differentiable functions, and let $f \in CS(f_1, \ldots, f_m)$. Then the following two statements hold:*
i) If x_0 is a non-critical point of f, then f and $f(x_0) + y_1$ are locally topologically equivalent at the point $(x_0, 0)$.
ii) If x_0 is a nondegenerate critical point of f, then f is locally topologically equivalent at $(x_0, 0)$ to a function of the form

$$f(x_0) + l(y_1, \ldots, y_k) - \sum_{i=k+1}^{k+\mu} y_i^2 + \sum_{j=k+\mu+1}^{n} y_j^2,$$

where $k = |\hat{I}(x_0)| - 1,\ l \in CS(y_1, \ldots, y_k, -\sum_{i=1}^{k} y_i)$, *and* μ *is the quadratic index of* f *at* x_0.
Moreover, the set of all m*-tuples* $(f_1 \ldots, f_m)$ *of* C^2*-functions such that all critical points of every continuous selection of the functions* $f_1 \ldots, f_m$ *are nondegenerate is an open and dense subset in the strong* C^2*-topology* (see [10], Chapter 7).

We call a collection of linear mappings $l_1, ..., l_m : \mathbb{R}^n \longrightarrow \mathbb{R}$ given by $l_i(x) = \langle a_i, x\rangle$ affinely independent if the vectors

$$\begin{pmatrix} a_1 \\ 1 \end{pmatrix}, \ldots, \begin{pmatrix} a_m \\ 1 \end{pmatrix} \in \mathbb{R}^{n+1}$$

are linearly independent. The following representation result is proved in [2].

Theorem 2.4 *Let* $l_1, \ldots, l_m \in \mathbb{R}^{n^*}$. *If* $l \in CS(l_1, \ldots, l_m)$, *then there exists a finite collection of index sets* $\{M_1, \ldots, M_r\}$ *with* $M_i \subseteq \{1, \ldots, m\}$ *and* $M_i \not\subseteq M_j$ *for* $i \neq j$ *such that*

$$l(x) = \max_{i \in \{1,\ldots,r\}} \min_{j \in M_i} l_j(x).$$

Moreover, if $l_1, \ldots, l_m$ *are affinely independent, then the collection of index sets is unique.*

If $l_1, \ldots, l_m$ are affinely independent, then the unique max-min representation

$$l(x) = \max_{i \in \{1,\ldots,r\}} \min_{j \in M_i} l_j(x)$$

of a function $l \in CS(l_1, \ldots, l_m)$ is called the max-min normal form of l. Using the laws for distributive lattices, this normal form can be transformed into a min-max representation

$$l(x) = \min_{k \in \{1,\ldots,s\}} \max_{j \in N_k} l_j(x)$$

where the index sets $N_i \subseteq \{1, \ldots, m\}$ are such that $N_i \neq\subseteq N_j$ for $i \neq j$. If $l_1, \ldots, l_m$ are affinely independent, then the collection of index sets $\{N_1, \ldots, N_s\}$ is again unique and the above representation is called the min-max normal form of l, [2].

For $l \in CS(l_1, \ldots, l_m)$ we denote by

$$K_+(l) = \{x \in \mathbb{R}^n \mid l(x) > 0 \} \quad \text{and} \quad K_-(l) = \{x \in \mathbb{R}^n \mid l(x) < 0 \},$$

the ascent- and descent cones of l. The max-min and min-max representations

$$l(x) = \max_{i \in \{1,\ldots,r\}} \min_{j \in M_i} l_j(x) = \min_{k \in \{1,\ldots,s\}} \max_{j \in N_k} l_j(x)$$

provide the following decomposition of $K_+(l)$ and $K_-(l)$ into convex cones:

$$K_+(l) = \bigcup_{i=1}^{r} K_+^i, \quad \text{and} \qquad K_-(l) = \bigcup_{k=1}^{s} K_-^k,$$

with $K_+^i = \{x \in \mathbb{R}^n \mid l_j(x) > 0,\ j \in M_i\}$ and $K_-^k = \{x \in \mathbb{R}^n \mid l_j(x) < 0,\ j \in N_k\}$.
If (ND1) holds at the origin for $l \in CS(l_1, \ldots, l_m)$, then the combinatorial structure of $K_+(l)$ and $K_-(l)$ is further characterized by the following result from [2].

Proposition 2.5 *Let $l_1, \dots, l_m \in I\!R^{n^*}$ and $l \in CS(l_1, \dots, l_m)$. If l satisfies (ND1) at the origin, then*

a) $\dim \mathrm{cl}K_+^i \cap \mathrm{cl}K_+^j \in \{0, n\}$ *and* $\dim \mathrm{cl}K_+^i \cap \mathrm{cl}K_+^j = n$ *if and only if* $|M_i \cup M_j| \leq n$.

a) $\dim \mathrm{cl}K_-^k \cap \mathrm{cl}K_-^l \in \{0, n\}$ *and* $\dim \mathrm{cl}K_-^k \cap \mathrm{cl}K_-^l = n$ *if and only if* $|N_k \cup N_l| \leq n$.

3 The Homotopy Type of the Lower-Level Set

Proposition 3.1 *Let $l \in CS(l_1, \dots, l_k)$ with $l_i \in I\!R^{n^*}$ such that $(ND1)$ holds at the origin, and let $q : I\!R^m \longrightarrow I\!R$ be a quadratic function of the form*

$$q(v) = -\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2.$$

Then the sets

$$\mathcal{L} = \{(u, v) \in I\!R^n \times I\!R^m \mid l(u) + q(v) \; < \; 0 \; \}$$

and

$$\mathcal{L}_0 = \{(u, v) \in I\!R^n \times I\!R^m \mid l(u) \; \leq \; 0 \; , q(v) \; \leq \; 0 \text{ and } (l(u), q(v)) \neq (0, 0) \; \}$$

are homotopy equivalent.

Proof: Let $i : \mathcal{L}_0 \hookrightarrow \mathcal{L}$ denote the inclusion map. We shall construct a retraction $R : \mathcal{L} \longrightarrow \mathcal{L}_0$ such that $R \circ i = \mathrm{Id}_{\mathcal{L}_0}$ and that $i \circ R \simeq \mathrm{Id}_{\mathcal{L}}$ ($i \circ R$ is homotopic to $\mathrm{Id}_{\mathcal{L}}$).

Note first that, by Proposition 2.5, the sets

$$K_l = \{u \in \mathrm{I\!R}^n \mid l(u) \; \leq \; 0\} \subset \mathrm{I\!R}^n \text{ and } K_q = \{v \in \mathrm{I\!R}^m \mid q(u) \; \leq \; 0\} \subset \mathrm{I\!R}^m$$

are locally finitely triangulable and contractible and are thus absolute retracts in view of [8] Chapter III, Corollary 6.4 and Proposition 7.2. Hence there exist retractions $\rho : \mathrm{I\!R}^n \longrightarrow K_l$ and $\sigma : \mathrm{I\!R}^m \longrightarrow K_q$ and thus the mapping $R : \mathcal{L} \longrightarrow K_l \times K_q$ defined by

$$R(u, v) = (\rho(u), \sigma(v))$$

is a retraction as well. By definition, $R \circ i = \mathrm{Id}_{\mathcal{L}_0}$ since for every $u \in K_l$ and every $v \in K_q$ we have $\rho(u) = u$ and $\sigma(v) = v$. Moreover, $R(\mathcal{L}) \subset \mathcal{L}_0$. In fact, if $(u, v) \in \mathcal{L}$ then either $l(u) < 0$ and hence $\rho(u) = u$ or $q(v) < 0$ and hence $\sigma(v) = v$. In both cases we obtain $(l(\rho(u)), q(\sigma(v))) \neq (0, 0)$.

Hence, it remains to be shown that $i \circ R$ is homotopic to the identity $\mathrm{Id}_{\mathcal{L}}$. To this end we consider the function $H : \mathcal{L} \times [0, 1] \longrightarrow \mathrm{I\!R}^n \times \mathrm{I\!R}^m$ defined by

$$H((u, v), t) = \begin{cases} (u \;,\; v) & : \quad l(u) \leq 0, \; q(v) \leq 0 \\ ((1-t) \cdot \rho(u) + t \cdot u \;,\; \eta(u, v, t) \cdot v) & : \quad l(u) > 0, \; q(v) < 0 \\ (\zeta(u, v, t) u \;,\; (1-t) \cdot \sigma(v) + t \cdot v) & : \quad l(u) < 0, \; q(v) > 0 \end{cases}$$

with

$$\eta(u,v,t) = \begin{cases} \sqrt{1 - \dfrac{\max\{l((1-t)\rho(u)+tu) - l(u), 0\}}{q(v)}} & , \ l(u) > 0 \\ & \quad q(v) < 0 \\ 1 & , \ l(u)\cdot q(v) = 0 \\ & \quad l(u)+q(v) < 0 \end{cases}$$

$$\zeta(u,v,t) = \begin{cases} 1 - \dfrac{(1-t^2)q(v) + 2t(t-1)(\sum\limits_{j=\mu+1}^{m} \sigma(v)_j v_j) - \sum\limits_{j=1}^{\mu} \sigma(v)_j v_j)}{l(u)} & , \ l(u) < 0 \\ & \quad q(v) > 0 \\ 1 & , \ l(u)\cdot q(v) = 0 \\ & \quad l(u)+q(v) < 0. \end{cases}$$

By definition H is continuous on the open set

$$\{(u,v) \in \mathcal{L} \times [0,1] \mid l(u) > 0, q(v) < 0 \ \text{ or } \ l(u) < 0, q(v) > 0\}.$$

Moreover, if $(u_n, v_n, t_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{L}\times[0,1]$ which converges to a point $(u_0, v_0, t_0) \in \mathcal{L} \times [0,1]$ not contained in the above set, then $H(u_n, v_n, t_n) \to (u_0, v_0)$. In fact, if $l(u_0) = 0$ and $q(v_0) < 0$, then $\rho(u_n) \to \rho(u_0) = u_0$ and thus $\eta(u_n, v_n, t_n) \to 1$, and if $q(v_0) = 0$ and $l(u_0) < 0$, then $\sigma(v_n) \to \sigma(v_0) = v_0$ and thus $\zeta(u_n, v_n, t_n) \to 1$. Note that $(l(u_0), q(v_0)) \neq (0,0)$ since $(u_0, v_0) \in \mathcal{L}$. Hence we conclude that H is continuous on $\mathcal{L} \times [0,1]$.
Next we show that $H(u,v,t) \in \mathcal{L}$ for every $t \in [0,1]$, provided $(u,v) \in \mathcal{L}$. If $l(u) \leq 0$ and $q(v) \leq 0$, then $H(u,v,t) = (u,v)$, so that it suffices to consider the case that at least one of the values $l(u)$ and $q(v)$ is positive. If, on the one hand, $l(u) > 0$, then $q(v) < 0$ since $(u,v) \in \mathcal{L}$ and thus $H(u,v,t) = ((1-t)\rho(u) + tu, \eta(u,v,t)v)$. Note that $\eta = \eta(u,v,t)$ solves the equation

$$\max\{l((1-t)\rho(u)+tu) - l(u), 0\} = (1-\eta^2(t)) \cdot q(v).$$

Hence

$$\begin{aligned} l((1-t)\rho(u)+tu) + q(\eta(u,v,t)v) &= l(u) + (l((1-t)\rho(u)+tu) - l(u)) + q(v) \\ &\qquad + (q(\eta(u,v,t)v) - q(v)) \\ &\leq l(u) + q(v) + \max\{l((1-t)\rho(u)+tu) - l(u), 0\} \\ &\qquad + (1-\eta(u,v,t)^2)\cdot q(v) \\ &= l(u) + q(v) < 0, \end{aligned}$$

and thus $H(u,v,t) \in \mathcal{L}$. If, on the other hand, $l(u) < 0$ and $q(v) > 0$, then $H(u,v,t) = (\zeta(u,v,t)u, (1-t)\sigma(v) + tv)$ and

$$l(\zeta(u,v,t)u) + q((1-t)\sigma(v)+tv) = \zeta(u,v,t)l(u) - \sum_{j=1}^{\mu}((1-t)\sigma(v)_j + tv_j)^2$$

$$
\begin{aligned}
& + \sum_{j=\mu+1}^{m} ((1-t)\sigma(v)_j + tv_j)^2 \\
= \;& \zeta(u,v,t)l(u) - (1-t)^2 \sum_{j=1}^{\mu} (\sigma(v)_j)^2 + (1-t)^2 \sum_{j=\mu+1}^{m} (\sigma(v)_j)^2 \\
& - \; t^2 \sum_{j=1}^{\mu} (\sigma(v)_j)^2 + t^2 \sum_{j=\mu+1}^{m} (\sigma(v)_j)^2 \\
& -2t(1-t) \sum_{j=1}^{\mu} \sigma(v)_j v_j + 2t(1-t) \sum_{j=\mu+1}^{m} \sigma(v)_j v_j \\
\leq \;& \zeta(u,v,t)l(u) + t^2 q(v) - 2t(1-t)(\sum_{j=1}^{\mu} \sigma(v)_j v_j + \sum_{j=\mu+1}^{m} \sigma(v)_j v_j),
\end{aligned}
$$

since

$$
q(\sigma(v)) = -\sum_{j=1}^{\mu} (\sigma(v)_j)^2 + \sum_{j=\mu+1}^{m} (\sigma(v)_j)^2 \leq 0.
$$

Hence we obtain

$$
\begin{aligned}
l(\zeta(u,v,t)u) + q((1-t)\sigma(v) + tv) \;\leq\;& \zeta(u,v,t)l(u) + t^2 q(v) - 2t(1-t)(\sum_{j=1}^{\mu} \sigma(v)_j v_j + \sum_{j=\mu+1}^{m} \sigma(v)_j v_j) \\
=\;& (\zeta(u,v,t) - 1)l(u) + l(u) + (t^2 - 1)q(v) \\
& +q(v) - 2t(1-t)(\sum_{j=1}^{\mu} \sigma(v)_j v_j + \sum_{j=\mu+1}^{m} \sigma(v)_j v_j) \\
=\;& l(u) + q(v) < 0,
\end{aligned}
$$

in view of the fact that $\zeta = \zeta(u,v,t)$ solves

$$
(\zeta - 1) \cdot l(u) = (1-t^2)q(v) + 2t(t-1)(-\sum_{j=1}^{\mu} \sigma(v)_j v_j + \sum_{j=\mu+1}^{m} \sigma(v)_j v_j).
$$

We conclude that the path $t \mapsto H((u,v),t)$ does not leave the set $\mathcal{L}$. We have thus proved the proposition since $H(u,v,0) = R(u,v)$ and $H(.,.,1) = Id_{\mathcal{L}}$. $\square$

Proposition 3.2 *Let $l \in CS(l_1,\ldots,l_m)$ with $l_1,\ldots,l_{n+1} \in I\!R^{n^*}$ and suppose (ND1) holds at the origin. Let $q : I\!R^m \longrightarrow I\!R$ be a quadratic function of the form*

$$
q(v) = -\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2
$$

Then the sets

$$
\mathcal{L}_0 = \{(u,v) \in I\!R^n \times I\!R^m \mid l(u) \leq 0 \,, q(v) \leq 0 \text{ and } (l(u), q(v)) \neq (0,0) \,\}
$$

and

$$
\mathcal{L}_0^* = \{(u,v) \in \mathcal{L}_0 \mid l(u) < 0 \,, \text{ for } u \neq 0 \text{ and } q(v) < 0 \text{ for } v \neq 0 \,\}
$$

are homotopy equivalent.

Proof: Let $l_S : S^{n-1} \to \mathbb{R}$ be defined by $l_S(u) = l(u)$ for every $u \in S^{n-1}$. Clearly l_S is piecewise smooth and $u \in S^{n-1}$ is a critical point of l_S if and only if there exists a scalar λ such that $\lambda u \in \partial l(u)$. Let $I \subseteq \{1, \ldots, n+1\}$ with $|I| \le n$. Note that, in view of (ND1),

$$\alpha_I = \min\{\| \sum_{i\in I} \lambda_i a_i \|_2 \mid \lambda_i \ge 0, \sum_{i \in I} \lambda_i = 1\} > 0.$$

Let $\alpha < \alpha_I$ for every $I \subseteq \{1, \ldots, n+1\}$ with $|I| \le n$. We show that $l_S^{-1}([-\alpha, \alpha])$ does not contain a critical point. Let $u \in S^{n-1}$ and $I(u) = \{i \mid l_i(u) = l(u)\}$. Note that $|I(u)| \le n$ since $l_i(u) = l_j(u)$ for every $i, j \in \{1, \ldots, n+1\}$ implies $u = 0$, in view of (ND1). If u is a critical point of l_S, then there exists a scalar λ and convex multipliers λ_i such that $\lambda u = \sum_{i \in I(u)} \lambda_i \nabla l_i(u)$. Note that $\lambda = l(u)$ in view of the definition of $I(u)$ and that $\| \sum_{i\in I(u)} \lambda_i \nabla l_i(u)\| \ge \alpha$. Thus $l(u) \ge \alpha$. We conclude from Proposition 1.2 and the homogeneity of the function l that there exists a continuous mapping

$$R : \mathrm{cl}K_-(l) \times [0,1] \longrightarrow \mathrm{cl}K_-(l)$$

with $R(u,t) < 0$ for every $t \in (0,1]$ and every $u \in \mathrm{cl}K_-(l) \backslash \{0\}$.

By assumption, the function q is given by:

$$q(v) = -\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2.$$

Now let $0 < \varepsilon < \varepsilon_0$ be sufficiently small and define

$$q_\varepsilon : \mathbb{R}^m \longrightarrow \mathbb{R} \ \text{ by } \ q_\varepsilon(v) = -(1+\varepsilon)\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2.$$

Then

$$K_-(q_\varepsilon) = \{x \in \mathbb{R}^m \mid q_\varepsilon(x) < 0\} \subsetneq K_-(q) = \{x \in \mathbb{R}^m \mid q(x) < 0\}.$$

In fact, the definition of q_ε implies that $q(u) < 0$ for every $u \in \mathrm{cl}K_-(q_\varepsilon) \setminus \{0\}$.

By integrating the vectorfield of inner normals on the level sets one obtains a family of homeomorphisms

$$d_\varepsilon : K_-(q) \longrightarrow K_-(q_\varepsilon)$$

which depends continuously on the parameter ε and gives a homotopy

$$D : \mathrm{cl}(K_-(q)) \times [0,1] \longrightarrow \mathrm{cl}(K_-(q), \qquad D(v,t) = d_{t\cdot\varepsilon_0}(v)$$

between $\mathrm{Id}_{K_-(q)}$ and d_{ε_0}.

In order to see that the cones

$$\mathcal{L}_0 = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m \mid l(u) \le 0, q(v) \le 0 \text{ and } (l(u), q(v)) \ne (0,0)\}$$

and

$$\mathcal{L}_0^* = \{(u,v) \in \mathbb{R}^n \times \mathbb{R}^m \mid l(u) < 0, \text{ for } u \ne 0 \text{ and } q(v) < 0 \text{ for } v \ne 0\}$$

are homotopy equivalent, not first that

$$\mathcal{L}_0^* \subset \mathcal{L}_0$$

and that

$$W : \mathcal{L}_0 \longrightarrow \mathcal{L}_0^*$$

with

$$W(u,v) = (R(u, q(v)), D(v, l(u)))$$

maps $\mathcal{L}_0$ onto $\mathcal{L}_0^*$ since for every $(u,v) \in \mathcal{L}_0$ we have $(l(u), q(v)) \neq (0,0)$. Moreover W is homotopic to the identity by the above construction. □

Proposition 3.3 *Let $l_1, \ldots, l_{n+1} \in I\!R^{n^*}$ be the following linear functions:*

$$l_i(u) = u_i \quad for \quad i \in \{1, ..., n\} \quad and \quad l_{n+1}(u) = -\sum_{j=1}^{n} u_j$$

and let $l \in CS(l_1, \ldots, l_m)$ be a continuous selections. Furthermore let $q : I\!R^m \longrightarrow I\!R$ be given by:

$$q(v) = -\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2$$

Then the sets

$$\mathcal{L}_0^* = \{(u,v) \in \mathcal{L}_0 \mid l(u) \;<\; 0 \;,\; for\ u \neq 0 \;\; and\ q(v) \;<\; 0\ for\ v \neq 0\ \}$$

and

$$\mathcal{L}_0^* \cap S^{n+m-1} = \{(u,v) \in \mathcal{L}_0^* \mid \langle u,u\rangle \;+\; \langle v,v\rangle = 1\ \}$$

are homotopy equivalent.

Proof: Since $(0,0) \in \mathbb{R}^{n+m}$ is not contained in $\mathcal{L}_0^*$, the retraction $x \mapsto \dfrac{x}{\|x\|}$ of $\mathbb{R}^{n+m} \setminus \{(0,0)\}$ to the $(n+m-1)$-sphere gives the homotopy equivalence between

$$\mathcal{L}_0^* = \{(u,v) \in \mathcal{L}_0 \mid l(u) \;<\; 0 \;,\; \text{for } u \neq 0 \;\; \text{and } q(v) \;<\; 0 \text{ for } v \neq 0\ \}$$

and

$$\mathcal{L}_0^* \cap S^{n+m-1} = \{(u,v) \in \mathcal{L}_0^* \mid \langle u,u\rangle \;+\; \langle v,v\rangle = 1\ \}$$

□

The join $X \star Y$ of two topological spaces X and Y, [3, 15], is defined to be the quotient space of the product $X \times Y \times [0,1]$ by the decomposition whose elements are the sets $\{\{x\} \times Y \times \{0\} \mid x \in X\} \cong X$ and $\{X \times \{y\} \times \{1\} \mid y \in Y\} \cong Y$ and the individual points of the set $X \times Y \times [0,1] \setminus (X \times Y \times \{0\}) \cup (X \times Y \times \{0\})$. With this definition we obtain the following homotopy equivalence (cf. [3] p. 468)

Proposition 3.4

$$\mathcal{L}_0^* \cap S^{n+m-1} \cong \begin{cases} \{u \in I\!R^n \mid l(u) \;<\; 0\ \} \star S^{\mu-1} \;\; for & : \;\; \mu \geq 1 \\ \{v \in I\!R^m \mid l(u) \;<\; 0\ \} & : \;\; \mu = 0 \end{cases}$$

Proof Assign

$$\mathcal{L}_0^* \cap S^{n+m-1} \longrightarrow \{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \} \star \{v \in \mathrm{I\!R}^m \mid q(u) \ < \ 0 \},$$

with

$$(u, v) \mapsto ((u, v) \ , \ \langle u, u \rangle).$$

Since the lower level set of the function q, i.e. $\{v \in \mathrm{I\!R}^m \mid q(u) \ < \ 0 \}$ is homotopy equivalent to the $\mu - 1$-sphere for $\mu \geq 1$, we obtain

$$\mathcal{L}_0^* \cap S^{n+m-1} \simeq \{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \} \star S^{\mu-1}.$$

For $\mu = 0$ we have

$$\mathcal{L}_0^* \cap S^{n+m-1} \simeq \{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \}$$

since $\{v \in \mathrm{I\!R}^m \mid q(u) \ < \ 0 \}$ is contractible. □

Proposition 3.5 *Let $l_1, \ldots, l_{n+1} \ \in I\!R^{n^*}$ be the following linear functions:*

$$l_i(u) = u_i \ \ \textit{for} \ \ \ i \in \{1, ..., n\} \ \ \textit{and} \ \ l_{n+1}(u) = -\sum_{j=1}^{n} u_j$$

and let $l \in CS(l_1, \ldots, l_m)$ be a continuous selections. Furthermore let $q : I\!R^m \longrightarrow I\!R$ be given by:

$$q(v) = -\sum_{j=1}^{\mu} v_j^2 + \sum_{j=\mu+1}^{m} v_j^2$$

and let

$$\mathcal{L} = \{(u, v) \in I\!R^n \times I\!R^m \mid l(u) \ + q(v) < \ 0 \}.$$

Then

$$H_p(\mathcal{L}) = \begin{cases} H_{p-\mu}(\{u \in I\!R^n \mid l(u) \ < \ 0 \}) & : \ p \ \geq \ 1, \mu \geq 0 \\ \\ Z\!\!\!Z & : \ p \ = \ 0, \ \ \mu > 0 \\ H_0(\{u \in I\!R^n \mid l(u) \ < \ 0 \}) & : \ p \ = \ 0, \ \ \mu = 0 \end{cases}$$

for the singular homology in $Z\!\!\!Z$.

Proof: Since

$$\begin{aligned} \mathcal{L} &= \{(u, v) \in \mathrm{I\!R}^n \times \mathrm{I\!R}^m \mid l(u) \ + q(v) < \ 0 \} \\ &\simeq \{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \} \star S^{\mu-1}, \end{aligned}$$

and a join with a sphere is a suspension, we obtain that, by an analogue of the Künneth formula (see for instance [3] Chapter IV, Problem 6.1),

$$H_p(\mathcal{L}) = H_{p-\mu}(\{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \})$$

for $p \geq 1$ and $\mu \geq 0$. For $p = 0$ and $\mu > 0$ the join of $\{v \in \mathrm{I\!R}^m \mid q(u) \ < \ 0 \}$ with the $\mu - 1$-sphere is pathwise connected, i.e. $H_0(\mathcal{L}) = Z\!\!\!Z$. For $p = 0$ and $\mu = 0$ the assertion follows from $\mathcal{L}_0^* \cap S^{n+m-1} \simeq \{u \in \mathrm{I\!R}^n \mid l(u) \ < \ 0 \}$. □

4 The Morse Inequalities

Let $l_1, \ldots, l_{n+1}$ be a collection of linear functions such that (ND1) holds. Then a continuous selection $l \in CS(l_1, \ldots, l_{n+1})$ has a unique min-max normal form

$$l(x) \;=\; \min_{k\in\{1,\ldots,s\}} \max_{j\in N_k} l_j(x),$$

[2]. The continuous selection l is thus described by the system of sets $\{(\bar{N}_k) \mid k \in \{1, ..., s\}\}$. Proposition 2.5 b) implies that $K_-(l)$ has the homotopy type of the simplicial complex $\mathcal{N}_l$ which is the nerve (see [15]) of the system of sets $\{(\bar{N}_k) \mid k \in \{1, ..., s\}\}$, i.e. the simplicial complex $\mathcal{N}_l$ which consists of the vertices $\bar{N}_1, ..., \bar{N}_s$ together with all symplices $\Delta(\bar{N}_{j_1}, ..., \bar{N}_{j_{k+1}})$ for which $\bar{N}_{j_1} \cap, ..., \cap \bar{N}_{j_{k+1}} \neq \emptyset$. Recall that $\bar{N}_j$ is the set $\{1, \ldots, n+1\} \backslash N_j$. There is a simple simplicial complex with the same nerve as $\mathcal{N}_l$ and only $(n+1)$ vertices. Take the standard simplex with $(n+1)$ vertices. Every proper subset of $\{1, \ldots, n+1\}$ defines a face of the simplex. Our complex is the union of the faces defined by the subsets $\bar{N}_j$. Note that the number of vertices of this comples does not exceed the dimension of the manifold plus one. Note that, by Proposition 2.5, for every $l \in CS(l_1, \ldots, l_{n+1})$ with

$$l(x) \;=\; \min_{k\in\{1,\ldots,s\}} \max_{j\in N_k} l_j(x)$$

the collection $\{(\bar{N}_k) \mid k \in \{1, ..., s\}\}$ satisfies

$$\bigcup_{k=1}^{s} N_k = \{1, ..., n+1\};$$

hence

$$\bigcap_{k=1}^{s} \bar{N}_k = \emptyset.$$

Thus all subcomplexes of the standard simplex appear as representations of $\mathcal{N}_l$, $l \in CS(l_1, \ldots, l_{n+1})$, except the homotopically trivial ones, where all simplices of the complex have a common point.

Following [13], the Morse inequalities are expressed in terms of relative homology of lower level sets. To this end, let M be a smooth manifold, $f_1, \ldots, f_m : M \longrightarrow \mathbb{R}$ be twice continuously differentiable functions, $f \in CS(f_1, \ldots, f_m)$, and let $x_0 \in M$ be a nondegenerate critical point of f. The Morse Lemma says that there is a system of coordinates (u, v) defined in a neighbourhood Q_{x_0} of x_0 such that in these coordinates the function f is of the form

$$f(x_0) + \min_{r\in\{1,\ldots,s\}} \max_{j\in N_r} l_j(u) - \sum_{i=k+1}^{k+\mu} v_i^2 + \sum_{j=k+\mu+1}^{m} v_j^2, \qquad (*)$$

where $k = |\hat{I}(x_0)| - 1$, every l_j is one of the functions $u \to u_i$ or $u \to -\sum_{i=1}^{k} u_i$, and μ is the quadratic index of f at x_0.

Consider the lower level sets

$$Mf_{\leq}(x_0) = \{x \in Q_{x_0} \mid f(x) \ \leq \ f(x_0) \ \} \ \text{ and } \ Mf_{<}(x_0) = \{x \in Q_{x_0} \mid f(x) \ < \ f(x_0) \ \}.$$

We shall calculate the local homology groups

$$H_p(f, x_0) \ = \ H_p(Mf_{\leq}(x_0), Mf_{<}(x_0))$$

with coefficients in $\mathbb{Z}$.

Theorem 4.1 *If x_0 is a nondegenerate critical point of f with the normal form $(*)$ and $Mf_<(x_0) \neq \emptyset$, then the singular homology groups can be computed in the following way:*

a) If $k = 0$ then:

$$H_p(f, x_0) \ = \begin{cases} \mathbb{Z} & : \ p = \mu \\ 0 & : \ \textit{otherwise.} \end{cases}$$

b) If $k \neq 0$ then:

$$H_p(Mf_<(x_0)) = \begin{cases} H_{p-\mu}(\mathcal{N}_l) & : \ p \ \geq \ 1, \\ \mathbb{Z} & : \ p \ = \ 0, \ \ \mu \neq 0, \\ H_0(\mathcal{N}_l) & : \ p \ = \ 0, \ \ \mu = 0, \end{cases}$$

where $\mathcal{N}_l$ is the simplicial complex with $k+1$ vertices and the maximal simplices $\bar{N}_1, \ldots, \bar{N}_s$.

Proof: Part a) can be shown as in the book of J.Milnor ([13], p. 29) and part b) follows immediately from Proposition 3.5 and Theorem 2.3. □

Theorem 4.2 *Let x_0 be a nondegenerate critical point of f.*

a) If f has a local minimum in $x_0 \in M$, i.e. $Mf_<(x_0) = \emptyset$, then

$$H_p(f, x_0) \ = H_p(Mf_{\leq}(x_0), \emptyset) \ = H_p(Mf_{\leq}(x_0)) = \begin{cases} \mathbb{Z} & : \ p = 0, \\ 0 & : \ p \ \neq \ 0. \end{cases}$$

b) If $Mf_<(x_0) \neq \emptyset$, then

$$H_p(f, x_0) \ = H_p(Mf_{\leq}(x_0), Mf_<(x_0)) \ = \begin{cases} H_{p-1}(Mf_<(x_0)) & : \ p \geq 2, \\ \mathbb{Z}^{\rho-1} & : \ p \ = 1, \\ 0 & : \ p \ = 0, \end{cases}$$

where ρ is the number of connected components of the set $Mf_<(x_0)$ (with $\mathbb{Z}^0 = 0$).

Proof: Since a nondegenerate local minimizer of f is strict, we obtain

$$H_p(f,x_0) \ = H_p(Mf_{\le}(x_0),\emptyset)) \ = H_p(Mf_{\le}(x_0)) = \begin{cases} \mathbb{Z} & : \quad p=0, \\ \\ 0 & : \quad p \ne 0, \end{cases}$$

Now suppose $Mf_<(x_0) \ne \emptyset$. Since, by Theorem 2.3, the set $Mf_{\le}(x_0)$ is contractible, the long exact homology sequence

$$\begin{array}{cccccccccc} \rightarrow & H_p(Mf_{\le}(x_0)) & \rightarrow H_p(Mf_{\le}(x_0), Mf_<(x_0)) & \rightarrow & H_{p-1}(Mf_<(x_0)) & \rightarrow & H_{p-1}(Mf_{\le}(x_0)) & \rightarrow \\ & \| & & & & & \| & \\ \rightarrow & 0 & \rightarrow H_p(Mf_{\le}(x_0), Mf_<(x_0)) & \rightarrow & H_{p-1}(Mf_<(x_0)) & \rightarrow & 0 & \rightarrow \end{array}$$

implies the isomorphism

$$H_p(Mf_{\le}(x_0), Mf_<(x_0)) \cong H_{p-1}(Mf_<(x_0))$$

for $p \ge 2$. For $p = 1$, the long exact sequence of the reduced homology (see [15]) implies that

$$H_1(Mf_{\le}(x_0), Mf_<(x_0)) \cong \tilde{H}_0(Mf_<(x_0)).$$

Hence, if the set $(Mf_<(x_0))$ has ρ connected components, then

$$H_1(Mf_{\le}(x_0), Mf_<(x_0)) = \mathbb{Z}^{\rho-1}$$

It remains to consider the case $p = 0$. Here we use again the long exact homology sequence:

$$0 \ \rightarrow \ H_1(Mf_{\le}(x_0), Mf_<(x_0)) \ \stackrel{\partial_*}{\rightarrow}$$

$$H_0(Mf_<(x_0)) \ \stackrel{i_*}{\rightarrow} \ H_0(Mf_{\le}(x_0)) \ \stackrel{j_*}{\rightarrow} \ H_0(Mf_{\le}(x_0), Mf_<(x_0)) \ \rightarrow \ 0$$

Since the homomorphism $i_* : \ H_0(Mf_<(x_0)) \ \longrightarrow \ H_0(Mf_{\le}(x_0))$ is always surjective, the homomorphism j_* maps $H_0(Mf_{\le}(x_0)) = \mathbb{Z}$ into 0; hence

$$H_0(Mf_{\le}(x_0), Mf_<(x_0)) \ = \ 0.$$

$\square$

Let us consider a continuous selection f of C^2-functions on a smooth n-dimensional manifold with compact lower level sets and suppose all critical points of f are nondegenerate. By definition, we have $H_p(f,x) = 0$ if $x \in M$ is not a critical point. Following J.Milnor [13] we consider the following inequality

$$\sum_{j=0}^{p}(-1)^{2p-j}\text{rank } H_{p-j}(M) \le \sum_{j=0}^{p}(-1)^{2p-j}(\sum_{x\in M}\text{rank } H_{p-j}(f,x)), \quad \text{for all } \ p \in \mathbb{N} \cup \{0\}.$$

This results in the following theorem.

Theorem 4.3 *a) (Inequalities of Betti-numbers)*

$$\textit{rank } H_p(M) \ \le \ \sum_{x\in M} \textit{rank } H_p(f,x), \quad p \in \mathbb{N} \cup \{0\}$$

b) (Equality of Euler-Characteristics)

$$\chi(M) \;=\; \sum_{x\in M} \chi(f,x)$$

with

$$\chi(f,x) \;=\; \sum_{p=0}^{n} (-1)^p rank\ H_p(f,x).$$

One may ask whether the use of upper level sets instead of lower level sets would lead to different Morse inequalities in the case of piecewise smooth functions. This is not the case as can be seen by applying Alexander duality. In fact, the upper level set is given by

$$K = \{(u,v) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid l(u) + q(v) > 0\} = \mathbb{R}^k \times \mathbb{R}^{n-k} \setminus \{(u,v) \mid l(u) + q(v) \le 0\},$$

and, analogously to Proposition 3.2, one can show that $\{(u,v) \in \mathbb{R}^k \times \mathbb{R}^{n-k} \mid l(u)+q(v) \le 0\}$ is homotopy equivalent to the set $\tilde{K} = \{(u,v)\mathbb{R}^k \times \mathbb{R}^{n-k} \mid l(u)+q(v) \;<\; 0\}$. Hence we have for $\mathbb{R}^n \setminus K$ and $\tilde{K}$ the same homology groups. Now consider the upper level set K as a subset of the one-point compactification of $\mathbb{R}^n$, i.e. $K \subset \mathbb{R}^n \cup \{\infty\} = S^{n+1}$. Alexander duality (see [3] Chapter VI, Corollary 8.6) gives rank $\tilde{H}_i(K) =$ rank $\tilde{H}_{n-i}(S^{n+1} \setminus K)$ for the reduced homology for all $i \in \mathbb{Z}$. Hence, we obtain the same Morse inequalities as for lower level sets.

Classical Morse inequalities are often used to obtain lower bounds for the number of nondegenerate critical points of a function on a manifold. In order to use our inequalities for the same purpose, one has to estimate the maximal possible contribution to the Betti numbers of one nondegenerate critical point. We come to the interesting problem: Given a simplicial complex C with $(k+1)$ vertices, what are the maximal possible values of Betti numbers $b_i =$ rank $H_i(C)$ and of the total Betti number $b = b_0 + \ldots + b_k$? Let us denote these maximal possible values by $B_i(k)$ and $B(k)$, respectively. We have $B_0(k) = k+1$, $B_{k-1}(k) = 1$, $B_j(k) = 0$ for $j > k-1$. Direct caculations give also $B(2) = 3$, $B(3) = 4$, $B(4) = 7$, $B_1(3) = 3$, $B_1(4) = 6$, $B_2(4) = 4$. A general formula for these numbers is presently not known.

5 Examples

We consider the two-dimensional case. Here we have the following types of nonsmooth normal-forms:

$$\begin{aligned}
f_1(x_1,x_2) &= \max\{x_1, x_2, -x_1-x_2\} \\
f_2(x_1,x_2) &= \min\{x_1, x_2, -x_1-x_2\} \\
f_3(x_1,x_2) &= \max\{x_1, \min\{x_2, -x_1-x_2\}\} \\
f_4(x_1,x_2) &= \max\{\min\{x_1, x_2\}, \min\{x_1, -x_1-x_2\}\} \\
f_5(x_1,x_2) &= \max\{\min\{x_1, x_2\}, \min\{x_2, -x_1-x_2\}\} \\
f_6(x_1,x_2) &= \max\{x_2, \min\{x_1, -x_1-x_2\}\} \\
f_7(x_1,x_2) &= \max\{-x_1-x_2, \min\{x_1, x_2\}\} \\
f_8(x_1,x_2) &= \max\{\min\{-x_1-x_2, x_1\}, \min\{x_2, -x_1-x_2\}\} \\
f_9(x_1,x_2) &= \max\{\min\{x_1, -x_1-x_2\}, \min\{x_2, -x_1-x_2\}, \min\{x_1, x_2\}\}
\end{aligned}$$

and the smooth normal formes:

$$g_1(x_1,x_2) = x_1^2 + x_2^2, \quad g_2(x_1,x_2) = -x_1^2 - x_2^2 \text{ and } g_3(x_1,x_2) = -x_1^2 + x_2^2$$

For a function $f \in \{g_1, f_1\}$ we have the following Betti-numbers for the local homology:

$$\text{(Max)} \qquad \beta_i = \text{rank } H_i(f,0) = \begin{cases} 0 & : \quad 0 \leq 1 \\ 1 & : \quad i = 2 \\ 0 & : \quad i \geq 3 \end{cases}$$

and for $f \in \{g_2, f_2\}$

$$\text{(Min)} \qquad \beta_i = \text{rank } H_i(f,0) = \begin{cases} 0 & : \quad i < 0 \\ 1 & : \quad i = 0 \\ 0 & : \quad i \geq 1 \end{cases}$$

For a function $f \in \{g_3, f_3, f_4, f_5, f_6, f_7, f_8\}$ we have the Betti-numbers:

$$\text{(SP1)} \qquad \beta_i = \text{rank } H_i(f,0) = \begin{cases} 0 & : \quad i \leq 0 \\ 1 & : \quad i = 1 \\ 0 & : \quad i \geq 2 \end{cases}$$

Finally, for the function f_9 we have:

$$\text{(SP2)} \qquad \beta_i = \text{rank } H_i(f_9,0) = \begin{cases} 0 & : \quad i \leq 0 \\ 2 & : \quad i = 1 \\ 0 & : \quad i \geq 1 \end{cases} .$$

Hence by classifying the normal-forms with respect to Betti-numbers, we have the above mentioned 4 types.

For the types (Max), (Min) and (SP1) we have the same situation as in the smooth case. First note that, by [2] Theorem 3.1, every function $h \in \{f_3, f_4, f_5, f_6, f_7, f_8\}$ is topologically equivalent to the function

$$f : [-1,+1] \times [-1,+1] \longrightarrow \mathbb{R} \text{ with } f(x_1, x_2) = |x_1| - |x_2|,$$

since all these functions have two ascent- and descent cones. This function f induces a function on the torus with a nonsmooth maximum and minimum and two nonsmooth saddle points of type (SP1). The saddle points are $(0,0)$ and the point which results from identifying $(1,1), (1,-1), (-1,-1), (-1,1)$, the maximum is the point which results from the identification of $(1,0)$ with $(-1,0)$ and the minimum is achieved at the point which results from the identification of $(0,1)$ with $(0,-1)$. The smooth analogon to this example is given by the function $\tilde{f}(x_1, x_2) = x_1^2 - x_2^2$.

The function f_9 has three ascend- and descent cones. In new coordinates u_1, u_2 defined by $x_1 = u_1 - u_2,\ x_2 = u_1 + u_2$ we have the representation

$$f_9(u_1, u_2) = \max\{\min\{u_1 - u_2, -2u_1\}, \min\{u_1 + u_2, -2u_1\}, \min\{u_1 - u_2, u_1 + x_2\}\}.$$

Note that

i) $f_9(0, u_2) = 0$ for all $v \in \mathbb{R}$

ii) $f_9(u_1, u_2) = f(u_1, -u_2)$ for all $u_1, u_2 \in \mathbb{R}$

iii) $f_9(-u_1, u_2) = -f(u_1, u_2)$ for all $u_1, u_2 \in \mathbb{R}$,

with the last identity resulting from the fact that

$$\begin{aligned} & \max\{\min\{u_1 - u_2, -2u_1\}, \min\{u_1 + u_2, -2u_1\}, \min\{u_1 - u_2, u_1 + x_2\}\} \\ = \ & \min\{\max\{u_1 - u_2, -2u_1\}, \max\{u_1 + u_2, -2u_1\}, \max\{u_1 - u_2, u_1 + x_2\}\}. \end{aligned}$$

Thus the function f_9 induces a function on the cylinder $\mathbb{R} \times [-1,+1]$ where the points $(u,1)$ and $(u,-1)$ are identified for every $u \in \mathbb{R}$. Moreover f_9 is antisymmetric with respect to the first variable and vanishes on the circle $\{0\} \times [-1,1]$. If we contract this circle to a point, then we can consider f_9 as function defined on a quadratic cone in $\mathbb{R}^3$ given by $u_1 u_2 = u_3^2$ which is antisymmetric with respect to the reflection at the origin.

Let us consider a smooth function $p : \mathbb{R}^2 \longrightarrow \mathbb{R}$ which is identical one on a disk with center $(0,0) \in \mathbb{R}^2$ and radius $r > 0$ and which vanishes outside the disk around $(0,0) \in \mathbb{R}^2$ with radius $r + \varepsilon$ for some $\varepsilon > 0$ and which satisfies $p(x_1, x_2) = p(-x_1, -x_2) = p(x_1, -x_2) = p(-x_1, x_2)$, e.g. the function

$$p_{r,\varepsilon} : \mathbb{R}^2 \longrightarrow \mathbb{R} \text{ with } p_{r,\varepsilon}(x_1, x_2) = 1 - \frac{\eta(\|(x_1, x_2)\| - r)}{\eta(\|(x_1, x_2)\| - r) + \eta(\varepsilon + r - \|(x_1, x_2)\|)}$$

where $\|(x_1, x_2)\|$ denotes the Euclidean norm of the vector (x_1, x_2) and

$$\eta(t) = \begin{cases} e^{-\frac{1}{t}} & t > 0 \\ 0 & t \leq 0. \end{cases}$$

Now the support of the function $f = f_9 \cdot p_{r,\varepsilon}$ is the disk $D = \{(x_1, x_2) \mid \|(x_1, x_2)\| \leq r + \varepsilon\}$ and f vanishes on the boundary ∂D of D. Since $f(-x_1, -x_2) = f_9(-x_1, -x_2) \cdot p_{r,\varepsilon}(x_1, x_2) = -f_9(x_1, x_2) \cdot p_{r,\varepsilon}(x_1, x_2)$ and f_9 has three ascend- and descent cones, we get a function on the two-sphere S^2 with three maxima, three minima and two saddle-points of type (SP2), if we identify ∂D to a point.

As a further example consider for $r + \varepsilon < 1$ the function

$$f : [-1,1] \times [-1,1] \longrightarrow \mathbb{R} \text{ with } f(x_1, x_2) = f_9(x_1, x_2) \cdot p_{r,\varepsilon}(x_1, x_2) + (1 - p_{r,\varepsilon}(x_1, x_2)) \cdot (|x_1| - |x_2|)$$

as a function on the torus. This is possible, since $p(x_1, x_2) = p(x_1, -x_2) = p(-x_1, x_2)$. This function has a maximum, a minimum and a saddle-point of type (SP1) which results from the second summand, a saddle-point of type (SP2), three maxima and three minima which result from the first summand, and five saddle-points of type (SP1) which result from the combination of both functions by the chosen partition of unity.

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