Abnormal Sub-Riemannian Geodesics: Morse Index and Rigidity

A.A. Agrachev[1] A.V. Sarychev[2]

Abstract. *Considering a smooth manifold M provided with a sub-Riemannian structure, i.e. with Riemannian metric and completely nonintegrable distribution, we set for two given points $q^0, q^1 \in M$ the problem of finding a minimal path out of those tangent to the distribution (admissible) and connecting these points. Extremals of this variational problem are called sub-Riemannian geodesics and we single out the abnormal ones which correspond to the vanishing Lagrange multiplier for the length functional. These abnormal geodesics are not related to the Riemannian structure but only to the distribution and, in fact, are singular points in the set of admissible paths connecting q^0 and q^1. Developing the Legendre-Jacobi-Morse-type theory of 2nd variation for abnormal geodesics we investigate some of their specific properties such as rigidity - isolatedness in the space of admissible paths connecting the two given points.*

1 Introduction

The paper deals with *abnormal sub-Riemannian geodesics.* Let us remind that a *sub-Riemannian structure* on a Riemannian manifold M is given by a *completely non-integrable* (or *completely non-holonomic*, or *possessing full Lie rank*) distribution $\mathcal{D}$ on M. A locally Lipschitzian path $q(\tau) \in W^1_\infty[0,T]$) ($W^1_\infty[0,T]$ denotes the space of Lipschitzian paths $\tau \to q(\tau)$ on M) is called *admissible* if its tangents ly in $\mathcal{D}$ for almost all $\tau \in [0,T]$. Given two points q^0 and q^1 we set a problem of finding *weakly* (or equivalently W^1_∞−locally) *minimal* admissible path connecting q^0 with q^1.

The problem looks like direct generalization of the classical Riemannian case, but in fact there is an essential difference. Namely the space of *all* locally Lipschitzian paths, which connect q^0 and q^1, has natural structure of Banach manifold. Critical points of the length functional on this manifold are Riemannian geodesics and all paths of minimal length are among them. On the contrary the space of *admissible* paths, which connect q^0 and q^1, is not in general a manifold; it may have singularities. These singularities correspond to so called *abnormal* sub-Riemannian geodesics, which do not depend on Riemannian structure on M and are completely determined by distribution $\mathcal{D}$.

The term 'abnormal' comes from optimization theory, since the problem of finding minimal admissible path can be obviously reformulated as a Lagrange problem of Calculus of Variations. The *extremals* of the last problem are *sub-Riemannian geodesics* and, in particular, *abnormal extremals,* with vanishing Lagrange multiplier for the (length) functional, are *abnormal sub-Riemannian geodesics.*

There was a lot of activity tended to elimination of abnormal sub-Riemannian geodesics. Preprint [18] of R.Montgomery lists several (given by different authors) false proofs of the fact, that a minimal admissible path should correspond to some normal sub-Riemannian geodesic.

[1]Leading Research Associate, Steklov Mathematical Institute, Russian Academy of Sciences, Vavilova ul. 42, 117966, Moscow, Russia; e-mail: andrei@agrachev.mian.su Partially supported by Russian Fund for Fundamental Research; grant No. 93-011-1728.

[2]Invited Associate Professor, Departamento de Matemática, Universidade de Aveiro, 3800, Aveiro, Portugal; e-mail: ansar@ua.pt On leave from: Institute for Control Sciences, Russian Academy of Sciences, Profsojuznaja ul. 65, 117342, Moscow, Russia. The work was partially carried out when the author was visiting Twente University, Enschede, The Netherlands under financial support of Dutch Organization for Pure Research (NWO)

The preprint contains also an important counterexample to this claim (see also [22]).

Main contribution of the paper is kind of Legendre-Jacobi-Morse-type theory of 2nd variation for abnormal geodesics and its implications. Starting with the definition of 2nd variation along an abnormal geodesic, we set 2nd-order necessary/sufficient minimality conditions for abnormal geodesics. The results have similar form to the classic Legendre-Jacobi minimality conditions of Calculus of Variations, but do not involving length functional, they are appearances of different phenomenon, which is 'degenerate' form of local minimality. Namely, the 2-nd order sufficient 'minimality' condition imply *rigidity* of abnormal geodesic path, which is isolatedness up to reparametrization of this path in $W^1_\infty-$topology in the space of all admissible paths, which connect given end-points. Therefore the 2-nd order necessary/sufficient minimality conditions are in fact necessary/sufficient rigidity conditions.

We go further and compute nullity and index of an abnormal geodesic, which are correspondingly dimension of the kernel and negative index of the 2nd variation along the abnormal geodesic. This in particular enables us to verify the 2-nd order rigidity conditions globally, on large time intervals. We use the Index and Nullity theorems to establish rigidity for several particular situations.

The paper is organized in following way. Section 2 contains preliminary material; of most importance for further presentation are some notations from chronological calculus and auxiliary results on symplectic geometry. In Section 3 we present Hamiltonian form of 'geodesic equation' and introduce some invariants of geodesics. In Section 4 we introduce 1st and 2nd variations along abnormal geodesics and define *Morse index* and *nullity.* Involving Goh and Generalized Legendre Condition along abnormal geodesics we derive (Theorem 4.4) a sufficient condition for smoothness of abnormal geodesic and announce (Theorems 4.1/4.8) necessary/sufficient conditions of rigidity. In Section 5 we introduce (Definition 5.1) *Jacobi curve in Lagrangian Grassmanian* for an abnormal geodesic and compute (Theorems 5.1 and 5.4) index and nullity of abnormal geodesics via symplectic invariants (Maslov-type indices) of the Jacobi curve. This enables us to establish (Theorem 5.5) local rigidity for abnormal geodesics meeting Goh and Strong Generalized Legendre Condition. In Section 6 we describe some class of distributions which do possess rigid abnormal geodesics (Theorem 6.1 and 6.2). In Section 7 we give more nice and simplified presentation of Legendre-Jacobi formalism for abnormal geodesics of 2-dimensional distributions. In Section 8 we investigate rigidity of trajectories for affine control systems (Theorems 8.4- 8.8). In Appendix (Section 9) we represent necessary/sufficient conditions (Theorems 9.1/ 9.5) for isolatedness of critical points of smooth mapping on critical level and use them to prove the necessary/sufficient conditions of rigidity for abnormal geodesics, which were established in the Section 4.

The presentation is self-contained, although we often refer to the paper [7], which deals with abnormal extremums for Lagrange problem of Calculus of Variations. One can find in that paper instructive analogies and details of some proofs.

In our work we were much inspired by a discussion on abnormal sub-Riemannian geodesics at the Conference 'Geometric Methods in Nonlinear Optimal Control' (Sopron, Hungary ,July 1991) and also by papers [10, 18] and discussions with M.Kawsky, R.Montgomery and H.J.Sussmann. The final version of this paper was prepared when the second author was visiting the Faculty of Applied Mathematics at Twente University, Enschede, The Netherlands ; he is grateful to the faculty staff and especially to H. Nijmeijer and A. van der Schaft for hospitality.

2 Preliminaries

In the paper we use notation and technical tools of chronological calculus developed by A.A. Agrachev and R.V. Gamkrelidze (see [5, 6]).

We will identify C^∞ diffeomorphisms $P : M \longrightarrow M$ with automorphisms of the algebra $C^\infty(M)$ of smooth functions on M: $\phi(\cdot) \longrightarrow P\phi = \phi(P(\cdot))$. The image of a point $q \in M$ under a diffeomorphism P will be denoted by $q \circ P$.

Vector fields on M are 1-st order differential operators on M or arbitrary derivations of the algebra $C^\infty(M)$, i.e. R-linear mappings $X : C^\infty(M) \longrightarrow C^\infty(M)$, satisfying the Leibnitz rule: $X(\alpha\beta) = (X\alpha)\beta + \alpha(X\beta)$. Value $X(q)$ of a vector field X at a point $q \in M$ lies in the tangent space $\mathcal{T}_q M$ to the manifold M at the point q. We denote by $[X^1, X^2]$ Lie bracket or commutator $X^1 \circ X^2 - X^2 \circ X^1$ of vector fields X^1, X^2. It is again a 1-st order differential operator and in local coordinates on M the Lie bracket can be presented as

$$[X^1, X^2] = [\sum_{i=1}^n X_i^1 \partial/\partial x_i, \sum_{i=1}^n X_i^2 \partial/\partial x_i] = \sum_{i=1}^n (\partial X_i^2/\partial x X^1 - \partial X_i^1/\partial x X^2)\partial/\partial x_i.$$

This operation introduces in the space of vector fields the structure of a Lie algebra denoted Vect M. For $X \in$ Vect$M$ we use the notation ad$X$ for the inner derivation of Vect M : $(\mathrm{ad}\, X)X' = [X, X'], \forall X' \in$ Vect M.

For a diffeomorphism P we use the notation $\mathrm{Ad}\, P$ for the following inner automorphism of the Lie algebra Vect M: $\mathrm{Ad}\, P X = P \circ X \circ P^{-1} = P_*^{-1} X$. The last notation stands for the result of translation of the vector field X by the differential of the diffeomorphism P^{-1}.

A flow on M is an absolutely continuous w.r.t. $\tau \in R$ curve $\tau \longrightarrow P_\tau$ in the group of diffeomorphisms Diff M, satisfying the condition $P_0 = I$ (where I is the identity diffeomorphism). We assume all time-dependent vector fields X_τ to be locally integrable with respect to τ. A time-dependent vector field X_τ defines an ordinary differential equation $\dot{q} = X_\tau(q(\tau)), q(0) = q^0$ on the manifold M; if solutions of this differential equation exist for all $q^0 \in M, \tau \in R$, then the vector field X_τ is called *complete* and defines a flow on M, being the unique solution of the (operator) differential equation:

$$dP_\tau/d\tau = P_\tau \circ X_\tau, P_0 = I. \tag{2.1}$$

This solution will be denoted by $P_t = \overrightarrow{\exp} \int_0^t X_\tau d\tau$, and is called (see [5, 6]) a *right chronological exponential* of X_τ. If the vector field $X_\tau \equiv X$ is time-independent, then the corresponding flow is denoted by $P_t = e^{tX}$.

We introduce also Volterra expansion (or Volterra series) for the chronological exponential. It is (see [5, 6]):

$$\overrightarrow{\exp} \int_0^t X_\tau d\tau \asymp I + \sum_{i=1}^\infty \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{i-1}} d\tau_i (X_{\tau_i} \circ \cdots X_{\tau_1}).$$

We will need only the terms of zero-, first- and second-order in this expansion, which are

$$\overrightarrow{\exp} \int_0^t X_\tau d\tau \asymp I + \int_0^t X_\tau d\tau + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 (X_{\tau_2} \circ X_{\tau_1}) + \cdots \tag{2.2}$$

For time-independent X one obtains

$$e^{tX} \asymp I + tX + (t^2/2) X \circ X + \cdots \tag{2.3}$$

One more tool from chronological calculus is a 'generalized variational formula'(see [5, 6] for its drawing):

$$\overrightarrow{\exp} \int_0^t (\hat{X}_\tau + X_\tau) d\tau = \overrightarrow{\exp} \int_0^t \hat{X}_\tau d\tau \circ \overrightarrow{\exp} \int_0^t \mathrm{Ad}(\overrightarrow{\exp} \int_t^\tau \hat{X}_\theta d\theta) X_\tau d\tau. \tag{2.4}$$

Applying the operator $\mathrm{Ad}(\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta)$ to a vector field Y and differentiating $\mathrm{Ad}(\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta) Y = (\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta) \circ Y \circ (\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta)^{-1}$ w.r.t. τ one comes to the equality (see [5, 6]):

$$\frac{d}{d\tau} \mathrm{Ad}(\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta Y) = \mathrm{Ad}(\overrightarrow{\exp} \int_0^\tau \hat{X}_\theta d\theta)\, \mathrm{ad}\, \hat{X}_\tau Y, \tag{2.5}$$

which is of the same form as (2.1). Therefore $\mathrm{Ad}(\overrightarrow{\exp}\int_0^\tau \hat{X}_\theta d\theta)$ can be presented as an operator chronologocal exponential $\overrightarrow{\exp}\int_0^t \mathrm{ad}\,\hat{X}_\theta d\theta$ which for a time-independent vector field $\hat{X}_\tau \equiv \hat{X}$ can be written as $e^{t\,\mathrm{ad}\,\hat{X}}$. These exponentials also admit Volterra expansions:

$$\overrightarrow{\exp}\int_0^t \mathrm{ad}\,X_\tau d\tau \asymp I + \sum_{i=1}^{\infty}\int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 \ldots \int_0^{\tau_{i-1}} d\tau_i (\mathrm{ad}\,X_{\tau_i} \circ \cdots \mathrm{ad}\,X_{\tau_1}) \asymp$$
$$\asymp I + \int_0^t \mathrm{ad}\,X_\tau d\tau + \int_0^t d\tau_1 \int_0^{\tau_1} d\tau_2 (\mathrm{ad}\,X_{\tau_2} \circ \mathrm{ad}\,X_{\tau_1}) + \cdots, \tag{2.6}$$

and

$$e^{t\,\mathrm{ad}\,X} \asymp I + t\,\mathrm{ad}\,X + (t^2/2)\,\mathrm{ad}\,X \circ \mathrm{ad}\,X + \cdots. \tag{2.7}$$

In this new notation the generalized variational formula (2.4) can be rerepresented as:

$$\overrightarrow{\exp}\int_0^t (\hat{X}_\tau + X_\tau) d\tau = \overrightarrow{\exp}\int_0^t \hat{X}_\tau d\tau \circ \overrightarrow{\exp}\int_0^t (\overrightarrow{\exp}\int_t^\tau \mathrm{ad}\,\hat{X}_\theta d\theta) X_\tau d\tau =$$
$$= \overrightarrow{\exp}\int_0^t (\overrightarrow{\exp}\int_0^\tau \mathrm{ad}\,\hat{X}_\theta d\theta) X_\tau d\tau \circ \overrightarrow{\exp}\int_0^t \hat{X}_\tau d\tau. \tag{2.8}$$

A *distribution* $\mathcal{D}$ on M is a subbundle of tangent bundle $\mathcal{T}M$, which we identify with the space of its sections. For a distribution $\dim \mathcal{D}_q$ does not change with $q \in M$.

Generalizations of distributions are *differential systems* or *distributions with singularities* [1] which are subbundles with nonconstant $\dim \mathcal{D}_q$. We call differential system any $C^\infty(M)-$submodule of $\mathrm{Vect}M$; then distributions correspond to projective $C^\infty-$modules. Locally one may treat germ of distribution as free module.

If $\mathcal{D}$ is a differential system, then taking $C^\infty-$modules generated by Lie brackets of order $\leq k,\ k = 1, \ldots,$ of the vector fields subjected to $\mathcal{D}$ one obtains an expanding sequence of differential systems:

$$\mathcal{D} \subseteq \mathcal{D}^2 = [\mathcal{D}, \mathcal{D}] \cdots \subseteq \mathcal{D}^k = [\mathcal{D}, \mathcal{D}^{k-1}] \subseteq \cdots.$$

For any $q \in M$ the sequence of subspaces

$$\mathcal{D}_q \subseteq \cdots \mathcal{D}_q^k \subseteq \mathcal{T}_q M$$

is called *flag of the differential system* $\mathcal{D}$ at the point $q \in M$, while the sequence $n_1(q) \leq \cdots n_k(q) \leq \cdots$, where $n_i(q) = \dim \mathcal{D}_q^i$, is called *growth vector of the differential system* $\mathcal{D}$ *at the point* q. Differential system is called *completely nonholonomic* or *having full Lie rank* at a point $q \in M$ if $\mathcal{D}_q^{\bar{k}} = \mathcal{T}_q M$ for some $\bar{k}$. Differential system is called *completely nonholonomic* or *having full Lie rank* if for some $\bar{k}$ $\mathcal{D}_q^{\bar{k}} = \mathcal{T}_q M$ for all $q \in M$.

If $\mathcal{D}$ is a distribution ($n_1(q) \equiv \mathrm{const}$), then still $\mathcal{D}^k$ may lack to be distributions (may have singularities), since the growth vector of a distribution in general changes with q. Distribution is called *regular* if its growth vector is constant for all q.

We also have to introduce some notions of symplectic geometry (see [8, 13, 17] for more details). A *symplectic structure* in an even-dimensional linear space Σ is defined by a nondegenerate bilinear skewsymmetric 2-form $\sigma(\cdot,\cdot)$. Two vectors $\xi_1, \xi_2 \in \Sigma$ are called skeworthogonal, written $\xi_1 \flat \xi_2$, if $\sigma(\xi_1, \xi_2) = 0$. If N is a subspace of Σ, let us denote by $N^\flat$ its skeworthogonal complement: $N^\flat = \{\xi \in \Sigma \mid \sigma(\xi, \nu) = 0, \forall \nu \in N\}$. Evidently $\dim N + \dim N^\flat = \dim \Sigma$. A subspace $\Gamma \subseteq \Sigma$ is called *isotropic*, when $\Gamma \subseteq \Gamma^\flat$, and *coisotropic*, when $\Gamma \supseteq \Gamma^\flat$. A subspace $\Lambda \subset \Sigma$ is called *Lagrangian plane*, when $\Lambda^\flat = \Lambda$. Such subspaces have dimension $\frac{1}{2}\dim \Sigma$. If Λ is a Lagrangian plane and Γ is isotropic, then it is easy to prove, that $(\Lambda \cap \Gamma^\flat) + \Gamma = (\Lambda + \Gamma) \cap \Gamma^\flat$ is a Lagrangian plane. We denote it by Λ^Γ.

[1] Not to be mixed with the differential systems determined by the differential forms; those have different kind of singularities

The symplectic group $Sp(\Sigma)$ is the group of those linear transformations of Σ, which preserve the symplectic form:

$$Sp(\Sigma) = S \in GL(\Sigma) \mid \sigma(S\xi_1, S\xi_2) = \sigma(\xi_1, \xi_2)\ \forall \xi_1, \xi_2 \in \Sigma.$$

The elements of this group are called symplectic transformations of Σ. The Lie algebra of the symplectic group is:

$$sp(\Sigma) = A \in gl(\Sigma) \mid \sigma(A\xi_1, \xi_2) = \sigma(A\xi_2, \xi_1)\ \forall \xi_1, \xi_2 \in \Sigma.$$

Let H be a real quadratic form on Σ and $d_\xi H$ be the differential of H at a point $\xi \in \Sigma$. Then $d_\xi H$ is a linear form on Σ which depends linearly on ξ. For every $\xi \in \Sigma$ there exists a unique vector $\vec{H}(\xi) \in \Sigma$ which satisfies equality $\sigma(\vec{H}(\xi), \cdot) = d_\xi H$. It is easy to show that the linear operator $\vec{H}$: $\Sigma \to \Sigma$ belongs to $sp(\Sigma)$, and the mapping $H \to \vec{H}$ is an isomorphism of the space of quadratic forms onto $sp(\Sigma)$. The differential equation $\dot{\xi} = \vec{H}(\xi)$ is called the linear Hamiltonian system corresponding to the quadratic Hamiltonian H.

Denote by $\mathcal{L}(\Sigma)$ the Grassmanian of Lagrangian subspaces of Σ. This is a smooth manifold of dimension $\frac{1}{8}\dim\Sigma(\dim\Sigma + 2)$.

Certainly symplectic transformations transform Lagrangian planes into Lagrangian ones, hence the symplectic group acts on $\mathcal{L}(\Sigma)$. It is easy to show that it acts transitively.

Let us consider a tangent space $\mathcal{T}_\Lambda\mathcal{L}(\Sigma)$, $\Lambda \in \mathcal{L}(\Sigma)$. To every quadratic form h on Σ there corresponds a linear Hamiltonian vector field $\vec{h}$ and a one-parameter subgroup $t \to e^{t\vec{h}}$ in $Sp(\Sigma)$. Let us consider the linear mapping

$$h \longrightarrow d(e^{t\vec{h}}\Lambda)/dt\mid_{t=0}$$

of the space of quadratic forms to $\mathcal{T}_\Lambda\mathcal{L}(\Sigma)$. This mapping is surjective and its kernel consists of all quadratic forms which vanish on Λ. Thus two different quadratic forms correspond to the same vector from $\mathcal{T}_\Lambda\mathcal{L}(\Sigma)$ if and only if the restrictions of these forms on Λ coincide. Hence we obtain a natural identification of the space $\mathcal{T}_\Lambda\mathcal{L}(\Sigma)$ with the space of quadratic forms on Λ.

A tangent vector $\eta \in \mathcal{T}_\Lambda\mathcal{L}(\Sigma)$ is called nonnegative if the corresponding quadratic form is nonnegative on Λ. An absolutely continuous curve Λ_τ $(\tau \in [0,T])$ in $\mathcal{L}(\Sigma)$ is called nondecreasing if the velocities $\dot{\Lambda}_\tau \in \mathcal{T}_{\Lambda_\tau}\mathcal{L}(\Sigma)$ are nonnegative for almost all $\tau \in [0,T]$.

Treating the action of symplectic group $Sp(\Sigma)$ on $\mathcal{L}(\Sigma)$ one can easily verify, that *pairs* of Lagrangian planes (Λ, Λ') have only one invariant w.r.t. this action: it is $\dim(\Lambda \cap \Lambda')$. For *triples* of Lagrangian planes, there are more invariants.

Let $\Lambda_1, \Lambda_2, \Lambda_3$ be Lagrangian planes. Let us present a vector $\lambda \in (\Lambda_1 + \Lambda_3) \cap \Lambda_2$ as a sum $\lambda = \lambda_1 + \lambda_3$ and consider on $(\Lambda_1 + \Lambda_3) \cap \Lambda_2$ properly defined quadratic form $\beta(\lambda) = \sigma(\lambda_1, \lambda_3)$. *Maslov index* of the triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ is signature of $\beta(\lambda)$. It is an invariant of the action of symplectic group.

In [1] a bit different invariant of a triple of Lagrangian planes $(\Lambda_1, \Lambda_2, \Lambda_3)$ was exploited for computation of Morse index for singular extremals.

Definition 2.1 *Consider the quadratic form $\beta(\lambda) = \sigma(\lambda_1, \lambda_3)$ with the domain $((\Lambda_1 + \Lambda_3) \cap \Lambda_2)/\bigcap_{i=1}^{3}\Lambda_i$. A sum $\frac{1}{2}\dim\ker\beta + \mathrm{ind}^-\beta$, where $\mathrm{ind}^-\beta$ is negative inertia index of β, is an invariant of the triple $(\Lambda_1, \Lambda_2, \Lambda_3)$ of Lagrangian planes. It is denoted by $\mathrm{ind}_{\Lambda_2}(\Lambda_1, \Lambda_3)$ and is called Maslov-type index.* $\square$

Let us note, that $\ker\beta = ((\Lambda_1 \cap \Lambda_2) + (\Lambda_2 \cap \Lambda_3))/\bigcap_{i=1}^{3}\Lambda_i$. We refer to [1] for a simple formula connecting this Maslov-type index with Maslov index of the triple and for the proof of the following 'triangle inequality':

$$\mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_3) \leq \mathrm{ind}_{\Lambda_0}(\Lambda_1, \Lambda_2) + \mathrm{ind}_{\Lambda_0}(\Lambda_2, \Lambda_3).$$

It also follows directly from the definition, that

$$\mathrm{ind}_{\Lambda_1}(\Lambda_1, \Lambda_3) = \frac{1}{2}\dim\ker\beta = \frac{1}{2}(\dim\Lambda_1 - \dim(\Lambda_1\cap\Lambda_3)). \tag{2.9}$$

A continuous curve $\Lambda(\tau) \in \mathcal{L}(\Sigma),\ 0 \leq \tau \leq 1$, is called *simple* if there exists $\Delta \in \mathcal{L}(\Sigma)$ such that $\Lambda(\tau) \cap \Delta = 0\ \forall \tau \in [0,1]$.

Lemma 2.1 *If* $\Lambda(\tau) \in \mathcal{L}(\Sigma),\ 0 \leq \tau \leq 1$, *is a simple nondecreasing curve in* $\mathcal{L}(\Sigma)$, *and* $\Pi \in \mathcal{L}(\Sigma)$, *then*

$$\mathrm{ind}_\Pi(\Lambda(0), \Lambda(1)) = \mathrm{ind}_\Pi(\Lambda(0), \Lambda(\tau)) + \mathrm{ind}_\Pi(\Lambda(\tau), \Lambda(1)),\ \forall \tau \in [0,1].\square$$

Lemma 2.2 *Let* $\Lambda^0, \Lambda^1 \in \mathcal{L}(\Sigma)$. *There exist* $\Delta \in \mathcal{L}(\Sigma)$ *and neighborhoods* $V^0 \ni \Lambda^0$, $V^1 \ni \Lambda^1$ *in* $\mathcal{L}(\Sigma)$ *such that whenever* $\Lambda \in V^0, \Lambda' \in V^1$ *and* $\dim(\Lambda\cap\Lambda') = \dim(\Lambda^0 \cap \Lambda^1)$ *then there exists a simple nondecreasing curve* $\Lambda(\tau),\ \tau \in [0,1]$ *such that* $\Lambda(0) = \Lambda, \Lambda(1) = \Lambda',\ \Lambda(\tau)\cap\Delta = 0\ \forall\tau \in [0,1]$. $\square$

Both Lemmas are proved in [1].

Definition 2.2 *Let* $\Lambda(t),\ 0 \leq t \leq T$, *be a nondecreasing curve in* $\mathcal{L}(\Sigma)$ *and* $0 = t_0 < t_1 < \cdots < t_l = T$ *are such, that the curves* $\Lambda(\cdot)\mid_{[t_i,t_{i+1}]},\ i = 0,\ldots l-1$, *are simple and* $\Pi \in \mathcal{L}(\Sigma)$. *The expression*

$$\mathrm{ind}_\Pi\Lambda(\cdot) = \sum_{i=0}^{l-1}\mathrm{ind}_\Pi(\Lambda(t_i), \Lambda(t_{i+1})) \tag{2.10}$$

is called Maslov index of the curve $\Lambda(t)$ *with respect to* Π. $\square$

It follows from the Lemma 2.1 that (2.10) does not depend on a choice of $t_1 < \cdots < t_{l-1}$. If the curve $\Lambda(t)$ is closed ($\Lambda(0) = \Lambda(T)$), then $\mathrm{ind}_\Pi\Lambda(\cdot)$ does not depend also on the choice of Π (cf. [1]).

3 Normal and Abnormal Geodesics. Rigidity.

The problem of finding minimal admissible path can be represented as following Lagrange problem of the Calculus of Variations with free final time:

$$\ell(T, u(\cdot)) = \int_0^T \langle G(q(\tau))u(\tau), G(q(\tau))u(\tau)\rangle^{1/2}d\tau \longrightarrow \min, \tag{3.1}$$

$$\dot{q} = G(q)u, q(0) = q^0, u \in R^r, \|u\| = 1, \tag{3.2}$$

$$q(T) = q^1. \tag{3.3}$$

Here $\langle\cdot,\cdot\rangle$ stays for the inner product in the tangent spaces T_qM; 'control parameter' u belongs to the $(r-1)-$dimensional unit sphere S^{r-1}; the controls $u(\tau)$ are measurable; $G(q) = (g^1(q), \ldots g^r(q))$ is a $r-$tuple of smooth vector fields, which form a basis of the distribution $\mathcal{D}$. Since our consideration regards a small neighborhood of a nonselfintersecting path on M, then such basis can always be chosen.

We investigate problem of *weak minimality*, i.e. whether a given time T and an admissible control $\hat{u}(\cdot)$ supply $(R \times L_\infty)$-local minimum for the problem (3.1)-(3.3).

Let us introduce classical 1-st-order necessary condition of weak optimality, for the Lagrange problem of Calculus of Variations. This is Euler-Lagrange equation in Hamiltonian form.

Theorem 3.1 *If a pair* $(T, \hat{u}(\cdot))$ *is weak minimizer for the problem (3.1)-(3.3), i.e. corresponding trajectory* $\hat{q}(\tau)$ $(\tau \in [0,T])$ *of (3.2) is* $W^1_\infty-$*locally minimal admissible path, then there exists a nonzero pair* $(\hat{\psi}_0, \hat{\psi}(\cdot))$, *where* $\hat{\psi}_0 \geq 0$ *is a constant and* $\hat{\psi}(\tau)$ *is an absolutely continuous*

covector-function on $[0,T]$, such that $\hat{\psi}(\tau) \in \mathcal{T}^*_{\hat{q}(\tau)}M$ and the 5-tuple $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}_0, \hat{\psi}(\cdot), T)$:
1) satisfies Hamiltonian system

$$\dot{q} = \partial H/\partial \psi, \ q(0) = q^0, \ q(T) = q^1, \tag{3.4}$$

$$\dot{\psi} = -\partial H/\partial q, \tag{3.5}$$

with a Hamiltonian

$$H(u, q, \psi_0, \psi) = \hat{\psi}_0 \langle G(q)u, G(q)u \rangle^{1/2} + \psi \cdot G(q)u; \tag{3.6}$$

2) meets stationarity condition

$$\frac{\partial H}{\partial u}|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}_0, \hat{\psi}(\tau))} v = 0, \ \forall v \in T_{\hat{u}(\tau)} S^{r-1}, \text{ for almost all } \tau \in [0,T], \tag{3.7}$$

and 'transversality condition'

$$H(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}_0, \hat{\psi}(\tau)) = 0, \text{ for almost all } \tau \in [0,T]. \ \square \tag{3.8}$$

Definition 3.1 *Sub-Riemannian geodesic is an extremal of the Lagrange problem (3.1)-(3.3), i.e. a 5-tuple $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}_0, \hat{\psi}(\cdot), T)$ meeting the conditions of the Theorem 3.1. Sub-Riemannian geodesic is called normal, if $\hat{\psi}_0 \neq 0$, and abnormal, if $\hat{\psi}_0 = 0$. The corresponding triple $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is called sub-Riemannian geodesic path.* $\square$

Remark. Obviously for any normal or abnormal sub-Riemannian geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}_0, \hat{\psi}(\cdot), T)$ its restriction $(\hat{u}(\cdot)|_{[0,t]}, \hat{q}(\cdot)|_{[0,t]}, \hat{\psi}_0, \hat{\psi}(\cdot)|_{[0,t]}, t)$ to a subinterval $[0,t] \subset [0,T]$ is also normal or abnormal sub-Riemannian geodesic correspondingly. ■

Remark. A geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ may enter different geodesics with different $\hat{\psi}_0, \hat{\psi}(\cdot)$. ■

Definition 3.2 *A corank of a geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is dimemsion of the space of pairs $(\hat{\psi}_0, \hat{\psi}(\cdot))$, which together with $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ satisfy Theorem 3.1.* $\square$

Definition 3.3 *A geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is called corank k abnormal geodesic path if the space of pairs $(0, \hat{\psi}(\cdot))$, which together with $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ satisfy the Theorem 3.1, is $k-$dimensional.* $\square$

Remark. One should take precautions, when determining corank of abnormal geodesic path, since in a $k-$dimensional linear space of pairs $(\hat{\psi}_0, \hat{\psi}(\cdot))$ there is $k-$ or $(k-1)$-dimensional subspace of pairs with vanishing ψ_0. Therefore it may happen, that corank k geodesic path is corank $(k-1)$ abnormal geodesic path. ■

Whenever geodesic is abnormal, then the length functional ℓ does not enter the minimality conditions, given by the Theorem 3.1. No surprise that corresponding geodesic paths have not too much to do with the sub-Riemannian metric and minimality of length. It turns out that they often exhibit a phenomenon called in [23] *rigidity.*

Definition 3.4 *An admissible path $q(\cdot)$ of the distribution $\mathcal{D}$ with end-points q^0 and q^1 is called rigid if it is isolated up to a reparametrization in the metric of W^1_∞ in the set $\mathcal{P}^{q^1}_{q^0}$ of all admissible paths, which connect q^0 and q^1.* $\square$

Rigid admissible paths are formally weakly minimal and analysis of the proof of the Theorem 3.1 shows, that the theorem is valid for the rigid paths as well; in addition one can take $\hat{\psi}_0 = 0$. This leads to

Proposition 3.2 *If an admissible path $(\hat{u}(\cdot), \hat{q}(\cdot))$ is rigid on $[0,T]$, then $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is an abnormal geodesic path.* $\square$

Remark. As it is known ([14]), admissible paths (without or with pregiven end-points) of a completely nonholonomic distribution $\mathcal{D}$ are dense in metric of C^0 in the space of all paths on M (correspondingly without or with pregiven end-points). Therefore an admissible path is never isolated in the metric of C^0, and hence strong ($=C^0-$local) minimality for sub-Riemannian geodesics is another deal. We are going to perform results on strong minimality of abnormal sub-Riemannian geodesics for 2-dimensional distributions in a forthcoming paper.■

To finish with the 1st-order condition given by the Theorem 3.1 let us note that in the abnormal case the Hamiltonian (3.6) degenerates into an 'abnormal' Hamiltonian

$$H = \psi \cdot G(q)u. \tag{3.9}$$

If we denote by $V^{\perp}_{\hat{u}(\tau)}$ the orthogonal complement to the vector $\hat{u}(\tau)$ in R^r, then the stationarity condition (3.7) for an abnormal geodesic takes form

$$\hat{\psi}(\tau) \cdot G(\hat{q}(\tau))v = 0, \ \forall v \in V^{\perp}_{\hat{u}(\tau)}, \ \forall \tau \in [0,T], \tag{3.10}$$

and (3.8) becomes:

$$H(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau)) = \hat{\psi}(\tau) \cdot G(\hat{q}(\tau))\hat{u}(\tau) = 0. \tag{3.11}$$

Together with (3.10) it implies orthogonality of $\hat{\psi}(\tau)$ to the distribution $\mathcal{D}$ at every point $\hat{q}(\tau)$:

$$\hat{\psi}(\tau) \cdot G(\hat{q}(\tau))v = 0, \ \forall v \in R^r, \ \forall \tau \in [0,T]. \tag{3.12}$$

4 Necessary/Sufficient Conditions for Rigidity of Abnormal Sub-Riemannian Geodesics

In the previous Section we have reduced the problem of finding minimal admissible (=tangent to the distribution $\mathcal{D}$) path between given points q^0 and q^1, to the Lagrange problem (3.1)-(3.3). We have formulated 1st-order necessary minimality condition saying that the solutions of this problem should be sought among geodesic paths. We have singled out the class of abnormal geodesics and defined what rigidity is. In this Section we are going to introduce 2nd variation and set 2-nd order necessary/sufficient conditions for rigidity of abnormal geodesic paths.

Let us start with definitions of first and second variations along an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. Everywhere in this Section we assume, that $\hat{u}(\cdot)$ is continuous (from the left) at T. Let us introduce a *(time × input)/state mapping* $F : R \times L^r_\infty[0,T] \longrightarrow M$, which maps a pair $(t, u(\cdot))$ into the point $q(t)$ of the trajectory $q(\cdot)$ of the system $\dot{q} = G(q)u(\tau), \ q(0) = q^0$. Obviously, $F(t, \hat{u}(\cdot)) = \hat{q}(\cdot)$ and $F(T, \hat{u}(\cdot)) = \hat{q}(T) = q^1$. We put

$$\ell(t, u(\cdot)) = \int_0^t \langle G(q(\tau))u(\tau), G(q(\tau))u(\tau)\rangle^{1/2} d\tau; \ \ell : R \times L^r_\infty[0,T] \to R.$$

A well known fact is that for $(T, \hat{u}(\cdot)) \in R \times L^r_\infty$ to be a minimizer for the Lagrange problem (3.1)-(3.3) it must be critical point of the mapping (ℓ, F). Indeed otherwise in virtue of the Implicit Function Theorem the system of equations

$$\ell(t, u(\cdot)) = \ell(T, \hat{u}(\cdot)) - \epsilon, \ F(t, u(\cdot)) = q^1,$$

is locally (in a neighborhood of $(T, \hat{u}(\cdot))$) solvable for any sufficiently small $\epsilon > 0$, and hence q^0 and q^1 can be connected by an admissible path of length $\ell(T, \hat{u}(\cdot)) - \epsilon < \ell(T, \hat{u}(\cdot))$. If a pair $(T, \hat{u}(\cdot))$ is critical point for the mapping (ℓ, F), i.e. the differential $(\ell', F')|_{(T,\hat{u}(\cdot))} : R \times L^r_\infty \to R \times \mathcal{T}_{q^1}M$ is nonsurjective, then there exists a pair $(\hat{\psi}_0, \hat{\psi}_T) \in R \times \mathcal{T}^*_{q^1}M$, which annihilates the image of $(\ell', F')|_{(T,\hat{u}(\cdot))}$:

$$\hat{\psi}_0 \ell' + \hat{\psi}_T F' \equiv 0. \tag{4.1}$$

This equality is equivalent to the statement of the Theorem 3.1 with $\hat{\psi}_T$ being the end-point value $\hat{\psi}(T)$ for the solution of the adjoint equation (3.5). If $\hat{\psi}_0 = 0$, then the functional ℓ does not enter both (4.1) and the Theorem 3.1. In this case the pair $(T, \hat{u}(\cdot))$ enters an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ or, equivalently, is critical point of the mapping F.

To study abnormal geodesics (=critical points of F) we have to invoke (first terms of) Taylor expansion for $F(t, u(\cdot))$. Let us present $F(t, u(\cdot))$ as chronological exponential (see Section 2 for the notation):

$$F(t, u(\cdot)) = q^0 \circ \overrightarrow{\exp} \int_0^t Gu(\tau)d\tau.$$

Putting $u(\tau) = \hat{u}(\tau) + v(\tau)$ and using the variational formula (2.4) we obtain

$$F(t, u(\cdot)) = q^0 \circ \overrightarrow{\exp} \int_0^t G(\hat{u}(\tau) + v(\tau))d\tau = q^0 \circ \overrightarrow{\exp} \int_0^t G\hat{u}(\tau)d\tau \circ \overrightarrow{\exp} \int_0^t Y_{t,\tau}v(\tau)d\tau, \quad (4.2)$$

where

$$Y_{t,\tau}v = \mathrm{Ad}\,\overrightarrow{\exp} \int_t^\tau G\hat{u}(\xi)d\xi Gv.$$

From the formula (2.5) it follows that

$$dY_{t,\tau}/dt = -\,\mathrm{ad}\,\hat{f}_t Y_{t,\tau}. \quad (4.3)$$

Putting

$$Y_\tau v = Y_{T,\tau}v = \mathrm{Ad}\,\overrightarrow{\exp} \int_T^\tau G\hat{u}(\xi)d\xi Gv, \quad (4.4)$$

we compute (compare with [7]) the first differential of F at the point $(T, \hat{u}(\cdot))$:

$$F'|_{(T,\hat{u}(\cdot))}(\delta\theta, u(\cdot)) = G(q^1)\hat{u}(T)\delta\theta + \int_0^T Y_\tau(q^1)u(\tau)d\tau, \ u(\tau) \in V^\perp_{\hat{u}(\tau)}. \quad (4.5)$$

If a pair $(T, \hat{u}(\cdot))$ is critical point of F, then $\mathrm{Im}F'|_{(T,\hat{u}(\cdot))} \neq \mathcal{T}_{q^1}M$, and there exists a nonzero covector $\hat{\psi}_T \in \mathcal{T}^*_{q^1}M$, which annihilates $\mathrm{Im}F'|_{(T,\hat{u}(\cdot))}$. This implies

$$\hat{\psi}_T \cdot G(q^1)\hat{u}(T) = 0, \quad (4.6)$$

and

$$\hat{\psi}_T \cdot \int_0^T Y_\tau(q^1)u(\tau)d\tau = 0,$$

for all $u(\cdot) \in L^r_\infty[0, T]$ such that $u(\tau) \in V^\perp_{\hat{u}(\tau)}$. In virtue of Dubois-Raymond Lemma the last equality implies:

$$\hat{\psi}_T \cdot Y_\tau(q^1)v = 0 \ \forall v \in V^\perp_{\hat{u}(\tau)}, \text{ for almost all } \tau \in [0, T]. \quad (4.7)$$

These conditions are equivalent to the conditions (3.10)-(3.11) of the Theorem 3.1 with the 'abnormal' Hamiltonian (3.9). Namely if we take the solution of the adjoint equation (3.5) with the end-point value $\hat{\psi}(T) = \hat{\psi}_T$, then the condition (4.7) is equivalent to the stationarity condition (3.7) and (4.6) implies, that the Hamiltonian $H = \psi Gu$, being constant along $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot))$, vanishes. The corank of abnormal geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ coincides with the corank of $F'|_{(T,\hat{u}(\cdot))}$.

Definition 4.1 *The first differential* $F'|_{(T,\hat{u}(\cdot))} : R \times L^r_\infty \to \mathcal{T}_{q^1}M$, *at a critical point* $(T, \hat{u}(\cdot))$ *is called first variation along abnormal geodesic path* $(\hat{u}(\cdot), \hat{q}(\cdot), T)$. *It is calculated according to the formula (4.5)* □

Now we introduce second variation along an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. It is Hessian, or quadratic differential of F, at the critical point $(T, \hat{u}(\cdot)) \in R \times L^r_\infty$ (see [9]). Choosing a function $\chi : M \longrightarrow R$, such that $d\chi|_{q^1} = \hat{\psi}_T$, let us consider a function $\phi(t, u(\cdot)) = \chi(F(t, u(\cdot)))$. Since $\hat{\psi}_T$ annihilates $\mathrm{Im}F'|_{(T,\hat{u}(\cdot))}$, then $(T, \hat{u}(\cdot))$ is critical point for this function.

Let us compute the quadratic term of Taylor expansion for $\phi(t, u(\cdot))$ at $(T, \hat{u}(\cdot))$. Appealing to the Volterra expansion (2.2) for right cronolological exponential, we derive

$$\phi''|_{(T,\hat{u}(\cdot))}(\delta\theta, u(\cdot)) = ((\int_0^T \int_0^\tau Y_\xi u(\xi)d\xi \circ Y_\tau(u(\tau)d\tau - \int_0^T [G\hat{u}(T)), Y_\tau^1 u(\tau)]d\tau +$$
$$+(G\hat{u}(T)) \circ (G\hat{u}(T))\frac{\delta\theta^2}{2} + (G\hat{u}(T))\delta\theta \circ \int_0^T Y_\tau u(\tau)d\tau)\chi)(q^1). \tag{4.8}$$

(When carrying the computation one should take into account the equalities (3.8), (4.3) and (4.7)).

When restricting the quadratic form (4.8) to the kernel of $F'|_{(T,\hat{u}(\cdot))}$, we are able to subtract from (4.8) a vanishing value of

$$\frac{1}{2}((G\hat{u}(T)\delta\theta + \int_0^T Y_\tau u(\tau)d\tau) \circ (G\hat{u}(T)\delta\theta + \int_0^T Y_\tau u(\tau)d\tau)\chi)(q^1),$$

and transform (4.8) into

$$\frac{1}{2}((\int_0^T [\int_0^\tau Y_\xi u(\xi)d\xi, Y_\tau u(\tau)]d\tau + [-G\hat{u}(T)\delta\theta, \int_0^T Y_\tau u(\tau)d\tau]\chi)(q^1).$$

The last expression does not depend on choice of χ but only on $\hat{\psi}_T = d\chi|_{q^1}$ and therefore we come to the

Definition 4.2 *The quadratic form*

$$2F''|_{(T,\hat{u}(\cdot))}[\psi_T](\delta\theta, u(\cdot)) = \hat{\psi}_T \cdot \int_0^T [-G\hat{u}(T)\delta\theta + \int_0^\tau Y_\xi u(\xi)d\xi, Y_\tau u(\tau)](q^1)d\tau, \tag{4.9}$$

whose domain is subspace of $R \times L_\infty^r$ defined by the condition

$$G(q^1)\hat{u}(T)\delta\theta + \int_0^T Y_\tau(q^1)u(\tau)d\tau = 0,\ u(\tau) \in V_{\hat{u}(\tau)}^\perp,\ \tau \in [0, T], \tag{4.10}$$

is called second variation along the abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. □

Definition 4.3 *Morse index of abnormal geodesic is negative index of the quadratic form* $(4.9) - (4.10)$*, i.e. maximal among the dimensions of the subspaces in its domain, on which the quadratic form is negative definite.* □

Definition 4.4 *Morse index of abnormal geodesic path is minimum of indices of those abnormal geodesics, which this geodesic path enters, or minimum of indices of quadratic forms $2F''|_{(T,\hat{u}(\cdot))}[\hat{\psi}_T]$ for all possible $\hat{\psi}_T \perp \mathrm{Im}F'|_{(T,\hat{u}(\cdot))}$.* □

We now set 2nd-order necessary rigidity condition for corank k abnormal geodesics paths. It follows from general necessary condition for isolatedness of critical point of smooth mapping on critical level. Formulation and proof of the general condition (Theorem 9.1) as well as the proof of the following Theorem 4.1 are given in the Appendix (Section 9). Corresponding result for corank 1 case was established in [7].

Theorem 4.1 (Necessary Rigidity Condition for Abnormal Geodesics) *For a corank k abnormal geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ to be rigid its index should not exceed $k-1$. In particular index of a rigid corank* 1 *abnormal geodesic path must vanish.* □

Generally rigidity is stronger than weak minimality. But whenever all geodesics, which a geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ enters, are abnormal, then the conditions of the Theorem 4.1 are necessary for weak minimality of the path. It follows from the Propositions 9.4 and 9.3 (see Appendix).

Proposition 4.2 (Necessary Minimality Condition for Abnormal Geodesics) *Let $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ be a corank k abnormal geodesic path, such that all geodesics it enters are abnormal. Then for the geodesic path to be weakly minimal its index should not exceed $k-1$.* □

It follows from the Theorem 4.1, that *finiteness of index is necessary for rigidity.* Therefore we are going to invoke Conditions which provide the finiteness for an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$.

Denoting again by $V^{\perp}_{\hat{u}(\tau)}$ the orthogonal complement to $\hat{u}(\tau)$ in R^r we introduce first of these conditions: for almost all $\tau \in [0, T]$

$$\partial/\partial u \frac{d}{d\tau} \partial H/\partial u|_{\hat{u}(\tau)}(v, w) = \hat{\psi}(\tau) \cdot [Gv, Gw](\hat{q}(\tau)) = 0 \ \forall v, w \in V^{\perp}_{\hat{u}(\tau)}. \tag{4.11}$$

In different context it was introduced by B.S. Goh in [12] and we call it *Goh necessary condition.*

Differentiating the identity (3.10) w.r.t. τ one obtains for almost all $\tau \in [0, T]$

$$0 = \frac{d}{d\tau} \partial H/\partial u|_{\hat{u}(\tau)}(w) = \hat{\psi}(\tau) \cdot [G\hat{u}(\tau), Gw](\hat{q}(\tau)) = 0 \ \forall w \in V^{\perp}_{\hat{u}(\tau)}. \tag{4.12}$$

and together with (4.11): for almost all $\tau \in [0, T]$

$$\hat{\psi}(\tau) \cdot [Gv, Gw](\hat{q}(\tau)) = 0 \ \forall v, w \in R^r. \tag{4.13}$$

We will also refer to the last condition as to Goh condition. This condition together with (3.12) implies, that at every point $\hat{q}(\tau)$ of rigid abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ the covector $\hat{\psi}(\tau)$ has to be orthogonal to $\mathcal{D}^2(\hat{q}(\tau)) = [\mathcal{D}, \mathcal{D}](\hat{q}(\tau))$, spanned by the vector fields from $\mathcal{D}$ and their Lie brackets of the 2nd order:

$$\hat{\psi}(\tau) \cdot Gv(\hat{q}(\tau)) = 0, \ \hat{\psi}(\tau) \cdot [Gv, Gw](\hat{q}(\tau)) = 0, \ \forall v, w \in R^r. \tag{4.14}$$

Another necessary condition, which is called (see [15, 4, 16]) *Generalized Legendre Condition*, is: for all $\tau \in [0, T]$

$$\partial/\partial u \frac{d^2}{d\tau^2} \partial H/\partial u|_{\hat{u}(\tau)}(v, v) = \gamma_\tau(v, v) = \hat{\psi}(\tau) \cdot [Gv, [G\hat{u}(\tau), Gv]](\hat{q}(\tau)) \geq 0 \tag{4.15}$$

(when computing this 4-th-order derivative we took into account the identity (4.13)); see the proof in [3, 2]).

We summarize the aforesaid in following

Proposition 4.3 (Necessary Goh and Generalized Legendre Conditions) *For an abnormal geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ to be rigid the Goh condition (4.13) and the Generalized Legendre Condition (4.15) have to hold for some abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$.* □

To set Jacobi-type conditions we need *Strong Generalized Legendre Condition.* It is (compare with (4.15)): for some $\beta > 0$ and for all $\tau \in [0, T]$

$$\gamma_\tau(v, v) = \hat{\psi}(\tau) \cdot [Gv[G\hat{u}(\tau), Gv]](\hat{q}(\tau)) \geq \beta \|v\|^2, \ \forall v \in V^{\perp}_{\hat{u}(\tau)}. \tag{4.16}$$

This last condition, which together with (4.13) is sufficient for finiteness of Morse index of an abnormal geodesic, is not only essential for its rigidity but also provides smoothness and in some cases uniqueness of the geodesic.

Theorem 4.4 (Regularity of Abnormal Geodesics) *Let Goh condition (4.11) and Strong Generalized Legendre Condition (4.16) hold along an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. Then the correspoding 'control' $\hat{u}(\tau)$ and the trajectory $\hat{q}(\cdot)$ are smooth on $[0, T]$. If in addition the vector space $[\mathcal{D}, \mathcal{D}])(q^0$ (correspondingly $[\mathcal{D}, \mathcal{D}](q^1)$) has codimension 1 in $\mathcal{T}_{q^0}M$ (correspondingly in $\mathcal{T}_{q^1}M$), then no other abnormal geodesic path, starting at q^0 (correspondingly, finishing at q^1) may satisfy Goh condition (4.11) and Generalized Legendre Conditions (4.15).*
□

Proof. Differentiating (4.13) w.r.t. τ we obtain

$$\hat{\psi}(\tau) \cdot [G\hat{u}(\tau), [Gv, Gw]](\hat{q}(\tau)) = 0, \ \forall v, w \in R^r, \forall \tau \in [0, T], \tag{4.17}$$

and, in particular,

$$\hat{\psi}(\tau) \cdot [G\hat{u}(\tau), [G\hat{u}(\tau), Gv]](\hat{q}(\tau)) = 0, \ \forall v \in R^r, \forall \tau \in [0, T]. \tag{4.18}$$

Hence the points $(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))$ of abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot))$ must ly in the subset of $S^{r-1} \times \mathcal{T}^*M$, defined by following system of relations:

$$\Phi(u, q, \psi) = \psi \cdot [Gu, [Gu, G]](q) = 0, \tag{4.19}$$

$$\Omega(u, q, \psi) = \psi \cdot [Gv, [Gu, Gv]](q) > 0, \ \forall v \in V^{\perp}_{\hat{u}(\tau)}. \tag{4.20}$$

Here $\Phi : R^r \times \mathcal{T}^*M \to (R^r)^*, \ \Omega : R^r \times \mathcal{T}^*M \to (R^r)^* \otimes (R^r)^*$.

The differential of the mapping Φ w.r.t. u at a point $(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))$ is:

$$\Phi'_u \Delta u = \hat{\psi}(\tau) \cdot [G\Delta u, [G\hat{u}(\tau), G]](\hat{q}(\tau)) + \hat{\psi}(\tau) \cdot [G\hat{u}(\tau)[G\Delta u, G]](\hat{q}(\tau)),$$

or since in virtue of (4.17) the last addend vanishes:

$$\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))} \Delta u = \hat{\psi}(\tau) \cdot [G\Delta u[G\hat{u}(\tau), G]](\hat{q}(\tau)). \tag{4.21}$$

Here $\Delta u \in V^{\perp}_{\hat{u}(\tau)} \simeq \mathcal{T}_{\hat{u}} S^{r-1}$, but if we substitute $\Delta u = \hat{u}(\tau)$ into (4.21), then in virtue of (4.18) $\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))} \hat{u}(\tau) = 0$. In virtue of (4.16) $\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))}$ is nonsingular on $V^{\perp}_{\hat{u}(\tau)}$ and hence the equation $\Phi(u, \hat{q}(\tau), \hat{\psi}(\tau)) = 0$ can be locally (in a small neighborhood of $(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))$) *uniquely* solved w.r.t. u, presenting u as a smooth function $u = u(\hat{q}(\tau), \hat{\psi}(\tau))$.

In fact the solution of the system (4.19)-(4.20) is *globally unique*, even more, there is no other solution $\bar{u}(q, \psi)$ of the equation (4.19) such that $\hat{\psi}(\tau) \cdot [Gv, [G\bar{u}, Gv]](\hat{q}(\tau))$ is nonnegative quadratic form on $V^{\perp}_{\bar{u}}$. Indeed let us assume, that $\Phi(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau)) = \Phi(\bar{u}, \hat{q}(\tau), \hat{\psi}(\tau)) = 0$. Then on the interval connecting $\hat{u}(\tau)$ with $\bar{u}$ there must be a point $u_\mu = \mu\hat{u}(\tau) + (1 - \mu)\bar{u}$ $(0 < \mu < 1)$ such that $\Phi'_u|_{(u_\mu, \hat{q}(\tau), \hat{\psi}(\tau))}(\hat{u}(\tau) - \bar{u}) = 0$ or, since $\Phi'_u|_{(u_\mu, \hat{q}(\tau), \hat{\psi}(\tau))}$ is linear w.r.t. u_μ,

$$(\mu\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))} + (1 - \mu)\Phi'_u|_{(\bar{u}, \hat{q}(\tau), \hat{\psi}(\tau))})(\hat{u}(\tau) - \bar{u}) = 0. \tag{4.22}$$

The left-hand side of (4.22) belongs to R^{r^*}; applying it to the vector $(\hat{u}(\tau) - \bar{u}) \in R^r$ and taking into account, that $\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))} \hat{u}(\tau) = \Phi'_u|_{(\bar{u}, \hat{q}(\tau), \hat{\psi}(\tau))} \bar{u} = 0$ we derive

$$0 = \mu\Phi'_u|_{(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau))}(\bar{u}, \bar{u}) + (1 - \mu)\Phi'_u|_{(\bar{u}, \hat{q}(\tau), \hat{\psi}(\tau))}(\hat{u}(\tau), \hat{u}(\tau)) = 0,$$

what may happen only if $\hat{u}(\tau) = \mp\bar{u}$. But if $\hat{u}(\tau) = -\bar{u}$, then the quadratic form $\hat{\psi}(\tau) \cdot [Gv[G\bar{u}, Gv]](\hat{q}(\tau))$ is negative definite. Hence $\hat{u}(\tau) = \bar{u}$.

Thus we have established, that for every τ the solution $\hat{u}(\tau)$ of the system of relations $\Phi(u, \hat{q}(\tau), \hat{\psi}(\tau)) = 0, \ \Omega(u, \hat{q}(\tau), \hat{\psi}(\tau)) \geq 0$ is globally unique. Then the corresponding implicit function $u(q, \psi)$, which is defined by the system (4.19)-(4.20), is continuous and hence smooth w.r.t. q, ψ and therefore $u(\hat{q}(\tau), \hat{\psi}(\tau))$ is smooth function of τ.

Assume, that $\mathrm{codim}[\mathcal{D}, \mathcal{D}](q^0) = 1$. Then, as we will prove now, there is no other geodesic path, starting at q^0 and meeting Goh and Ceneralized Legendre Conditions (4.11) and (4.15).

Assume on the contrary that there is another geodesic $(\bar{u}(\cdot), \bar{q}(\cdot), \bar{\psi}(\cdot), T)$ which starts at q^0 and meets the conditions (4.13) and (4.15). Then $\bar{\psi}(0) = k\hat{\psi}(0)$, or since the geodesic equaions and the conditions (4.13) and (4.15) are homogeneous in $\hat{\psi}$, we may think, that $\bar{\psi}(0) = \hat{\psi}(0)$. We have already established existence of a unique smooth function $u(q, \psi)$ defined on some neighborhood $\mathcal{W}$ of the point $(\hat{q}(0), \hat{\psi}(0))$ such that: $\hat{u}(\tau) = u(\hat{q}(\tau), \hat{\psi}(\tau)), \ \bar{u}(\tau) = u(\bar{q}(\tau), \bar{\psi}(\tau))$. That means, that $(\hat{q}(\cdot), \hat{\psi}(\cdot))$ and $(\bar{q}(\cdot), \bar{\psi}(\cdot))$ are locally (in $\mathcal{W}$) solutions of the same Hamiltonian system with the same starting points. Hence they coincide in $\mathcal{W}$. Standard reasoning proves, that the set of those t, for which $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot))$ and $(\bar{q}(\cdot), \bar{q}(\cdot), \bar{\psi}(\cdot))$ coincide, is closed and open in $[0, T]$ and hence is $[0, T]$ itself. The same reasoning is applicable to the geodesic paths finishing at the point q^1. ■

Now we are going to set 2nd-order *sufficient* rigidity condition for abnormal geodesics. It involves the introduced above Goh and Generalized Strong Legendre Conditions, which provide for the second variation (4.9)-(4.10) *weak positive definiteness* on some subspace of finite codimension in the domain (4.10) (see [1]). To put it in a strict way, let us note, that if Goh condition (4.11) holds, then the quadratic form (4.9)-(4.10) can be ([1, Lemma 3.8]) extended by continuity onto subspace of finite codimension in $R \oplus H^r_{-1}[0, T]$; the subspace is determined by the condition (4.10). The notation $H^r_{-1}[0, T]$ stays for Sobolev space of order -1, which is dual space to the space $H^r_1[0, T]$ of absolute-continuous functions with square integrable derivatives. The space $L^r_2[0, T]$ is densely embedded into $H^r_{-1}[0, T]$. For any function $u(\cdot) \in L^r_2[0, T] \subset H^r_{-1}[0, T]$, whose primitive is $v(\cdot) = \int_0^{\cdot} u(\tau)d\tau$, the H_{-1}-norm of $u(\cdot)$ can be defined as: $\|u(\cdot)\|_{-1} = (|v(T)|^2 + \|v(\cdot)\|^2_{L_2})^{1/2}$. There is a direct estimate

$$\|u(\cdot)\|_{-1} \leq (T + \sqrt{T})\|u(\cdot)\|_{L_2}.$$

Let us also note, that fast-oscillating functions have small $H_{-1}-$norms. Thus for a finite interval $[0, T]$: $\|\sin \tau/\delta\|_{-1} = O(\delta)$, for $\delta \to 0$, while $\|\sin \tau/\delta\|_{L_2} = T/2 + O(\delta)$, for $\delta \to 0$.

We define weak positive definiteness of a quadratic form in $L^r_2[0, T]$ as its positive definiteness w.r.t. the norm of $H^r_{-1}[0, T]$. We refer to [1] for the proof of the following

Proposition 4.5 *If Goh condition (4.11) and Generalized Strong Legendre Condition (4.16) both hold along abnormal geodesic, then the extension of the second variation (4.8) is weak positive definite on some subspace of finite codimension in its domain which is linear subspace of $R \times H^r_{-1}[0, T]$ determined by the condition (4.10). On this subspace the second variation admits lower estimate:*

$$2F''|_{(T, \hat{u}(\cdot))}[\psi_T](\delta\theta, u(\cdot)) \geq c(\|u(\cdot)\|^2_{-1} + \delta\theta^2). \ \square$$

Corollary 4.6 *Under the conditions of the Proposition the extended second variation admits on some subspace of finite codimension in its domain a lower estimate*

$$2F''|_{(T, \hat{u}(\cdot))}[\psi_T](\delta\theta, u(\cdot)) \geq c(\|v(\cdot)\|^2_{L_2} + \delta\theta^2),$$

where $v(\cdot) = \int_0^{\cdot} u(\tau)d\tau$. □

Everywhere below we assume that Goh condition (4.11) and Generalized Strong Legendre condition (4.16) hold along geodesics we deal with. This implies, that the negative indices of the second variation $(4.9) - (4.10)$ and of its extension onto (subspace of) $R \times H^r_{-1}[0, T]$ are finite and coincide.

Definition 4.5 *Nullity of an abnormal sub-Riemannian geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ is the dimension of the kernel of the second variation* $(4.9) - (4.10)$ *in* $R \times H^r_{-1}[0, T]$. □

It turns out, that in fact under the assumptions, we have made, the kernel 'is almost contained in $R \times L^r_\infty$, namely it is contained in $R \times (L^r_\infty \oplus H^r_{-1}[0] \oplus H^r_{-1}[T])$, where $H^r_{-1}[0]$ and $H^r_{-1}[T]$ consist of R^r-valued Dirac measures located at 0 and T correspondingly. Following fact was established in [1].

Proposition 4.7 *Under Generalized Strong Legendre Condition (4.16) and Goh condition (4.13) the kernel of the second variation is contained in $R \times (L^r_\infty \oplus H^r_{-1}[0] \oplus H^r_{-1}[T])$, moreover for an element $(\theta, u(\cdot))$ of the kernel its second component $u(\cdot)$ is C^∞ on $(0, T)$.* □

Now we are able to set Sufficient Rigidity Condition for abnormal geodesics.

Theorem 4.8 (Sufficient Condition of Rigidity for Abnormal Geodesics) *If the second variation along abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ is weak positive definite, then the geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is rigid, i.e. isolated up to a reparametrization in the topology of W^1_∞ in the set of admissible paths, which connect q^0 and q^1.* □

Corollary 4.9 *If Goh condition (4.11) and Generalized Strong Legendre Condition (4.16) hold along abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ and its Morse index and nullity both vanish, then the geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is rigid.* □

The two results follow from general sufficient condition for isolatedness of critical points of smooth mappings on critical levels; formulation and proof of the general condition (Theorem 9.5) as well as the proof of the Theorem 4.8 are to be found in the Appendix (Section 9).

5 Morse Index and Nullity of Abnormal Sub-Riemannian Geodesics

In the previous Section we have set necessary (Theorem 4.1) and sufficient (Theorem 4.8) conditions for the rigidity of abnormal geodesics. The corresponding statements involve Morse index and nullity, and in this Section we are going to compute Morse index and nullity for an abnormal geodesic. The scheme of the computation is in many aspects similar to the one presented in [7] for abnormal extremals of Lagrange Problem of Calculus of Variations. Refering to that paper for more details, we still provide a selfcontained exposition. The readers are referred to the Section 2 for notions and facts from symplectic geometry.

We start with the computation of the Morse index. To this purpose we introduce symplectic representation of the second variation $(4.9)-(4.10)$ along abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$.

Let us put

$$W = \text{span}\{\{G(q^1)\hat{u}(T)\} \cup \{Y_\tau(q^1)v | \tau \in [0,T],\ v \in V^\perp_{\hat{u}(\tau)}\}\}, \tag{5.1}$$

where Y_τ is defined by (4.4). Evidently $W \subset \mathcal{T}_{q^1}M$ coincides with an image $\text{Im}F'|_{(T,\hat{u}(\cdot))}$ of the first variation (4.5). It follows from (4.6)-(4.7), that $\hat{\psi}_T$ annihilates W, and codim $W = k$ is corank of the abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$.

Taking the space $\mathcal{E}_W$ of the vector fields, whose values at q^1 ly in W, let us consider a skewsymmetric bilinear form on $\mathcal{E}_W$:

$$\hat{\psi}_T \cdot [X, X'](q^1),\ \forall X, X' \in \mathcal{E}_W. \tag{5.2}$$

This form has kernel of finite codimension in $\mathcal{E}_W$, which is defined by equalities:

$$X(q^1) = 0; \hat{\psi}_T \cdot (\partial X/\partial \xi)(q^1) = 0, \forall \xi \in W.$$

Taking the quotient of $\mathcal{E}_W$ w.r.t. this kernel, one obtains on the finite-dimensional quotient space Σ a (induced from (5.2)) *nondegenerate* skewsymmetric bilinear form $\sigma(\cdot,\cdot)$. This form defines symplectic structure on Σ. Direct calculation gives us $\dim \Sigma = 2 \dim W = 2(n-k)$. We denote by $\underline{X}$ the image of an $X \in \mathcal{E}_W$ under the canonical projection $\mathcal{E}_W \longrightarrow \Sigma$.

Choosing local coordinates $(x_1, \ldots x_n) : \mathcal{O} \longrightarrow R^n$ on some neghborhood $\mathcal{O}$ of q^1 in M in such a way that $x_i(q^1) = 0,\ (i = 1, \ldots n)$ and the subspace W is defined by the equalities $x_1 = \cdots = x_k = 0$ while $\hat{\psi}_T = (\psi_1, \ldots, \psi_k, 0, \ldots 0)$, we may represent the canonical projection $X \to \underline{X}$ as:

$$X = \sum_{i=1}^n X_i(x)\partial/\partial x_i \to \underline{X} =$$
$$(X_{k+1}(0), \ldots X_n(0), \partial(\sum_{i=1}^k \psi_i X_i)/\partial x_{k+1}|_0, \ldots \partial(\sum_{i=1}^k \psi_i X_i)/\partial x_n|_0). \tag{5.3}$$

The symplectic form $\sigma(\underline{X},\underline{Y})$ can be then represented as:

$$\sigma(\underline{X},\underline{Y}) = \sum_{j=k+1}^{n} (X_j(0)\partial(\sum_{i=1}^{k}\psi_i Y_i)/\partial x_j|_0 - Y_j(0)\partial(\sum_{i=1}^{k}\psi_i X_i)/\partial x_j|_0).$$

Let us denote by Π the image under the canonical projection of the space of those vector fields, which vanish at q^1. Since the Lie bracket of two vanishing at q^1 vector fields also vanish at q^1, then Π is Lagrangian plane.

Instead of notations $\underline{Y}_\tau$ and $\underline{G\hat{u}(T)}$ for the images of the vector fields Y_τ and $G\hat{u}(T)$ under the canonical projection $\mathcal{E}_W \to \overline{\Sigma}$ we use below Υ_τ and $\hat{g}$ correspondingly. According to the introduced above definitions of $\sigma(\cdot,\cdot)$ and Π, we may represent the second variation (4.9)-(4.10) as:

$$2F''|_{[T,\hat{u}(\cdot),\psi_T]}(\delta\theta, u(\cdot)) = \int_0^T \sigma(-\hat{g}\delta\theta + \int_0^\tau \Upsilon_\xi u(\xi)d\xi, \Upsilon_\tau u(\tau))d\tau, \tag{5.4}$$

and its domain as:

$$\{(\delta\theta, u(\cdot)) \in R \times L_\infty^r | \hat{g}\delta\theta + \int_0^T \Upsilon_\tau u(\tau)d\tau \in \Pi\}. \tag{5.5}$$

Under new notations the Goh condition (4.13) and Strong Generalized Legendre Condition (4.16) take form:

$$\sigma(\Upsilon_\tau u, \Upsilon_\tau v) = 0,\ \forall u, v \in R^r,\ \forall \tau \in [0,T], \tag{5.6}$$

and

$$\gamma_\tau(u,u) = \sigma(\dot{\Upsilon}_\tau u, \Upsilon_\tau u) \geq \beta|u|^2, \forall u \in V^\perp_{\hat{u}(\tau)}, \tag{5.7}$$

correspondingly.

Now we will transform the formulae (5.4)-(5.5) for the second variation, representing it as a quadratic form in $\delta\theta$ and $v(\cdot) = \int_0^\cdot u(\xi)d\xi$ instead of $\delta\theta, u(\cdot)$. To this end let us integrate (5.4) by parts in such a way, that $u(\cdot)$ is integrated:

$$2F''|_{[T,\hat{u}(\cdot),\psi_T]}(\delta\theta, u(\cdot)) = \int_0^T \sigma(-\hat{g}\delta\theta + \int_0^\tau \Upsilon_\xi \underbrace{u(\xi)d\xi}_{dv}, \Upsilon_\tau u(\tau))d\tau =$$

$$\int_0^T \sigma(-\hat{g}\delta\theta + \Upsilon_\tau v(\tau), \Upsilon_\tau \underbrace{u(\tau))d\tau}_{dv} - \int_0^T \sigma(\int_0^\tau \dot{\Upsilon}_\xi v(\xi)d\xi, \Upsilon_\tau u(\tau))d\tau =$$

$$\int_0^T \sigma(\dot{\Upsilon}_\tau v(\tau), \Upsilon_\tau v(\tau))d\tau + \int_0^T \sigma(\hat{g}\delta\theta + \Upsilon_T v(T) + \int_0^\tau \dot{\Upsilon}_\xi v(\xi)d\xi, \dot{\Upsilon}_\tau v(\tau))d\tau.$$

When proceeding with this computation we took into account that

$$\sigma(\Upsilon_\tau v(\tau), \Upsilon_\tau u(\tau)) = 0,\ \sigma(\hat{g}, \Upsilon_T v(T)) = \sigma(\hat{g}, \underline{Gv(T)}) = 0$$

in virtue of Goh condition (5.6).

The domain of the second variation is:

$$\{(\delta\theta, v(T), v(\cdot)) | \hat{g}\delta\theta + \underline{Gv(T)} - \int_0^T \dot{\Upsilon}_\tau v(\tau)d\tau \in \Pi\}. \tag{5.8}$$

Let us put

$$\Gamma = \mathrm{span}\{\underline{Gv} | v \in R^r\},\ \Gamma_\varrho = \mathrm{span}\{\underline{Gv} | v \in V^\perp_{\hat{u}(T)}\}\ (\Gamma_\varrho \subset \Gamma). \tag{5.9}$$

In virtue of Goh condition (5.6) $\sigma(\underline{Gv}, \underline{Gw}) = \sigma(\Upsilon v, \Upsilon w) = 0$ and therefore Γ (and Γ_ϱ) is isotropic subspace of $\Sigma :\ \Gamma \subseteq \Gamma^\flat$.

Following [1] we introduce now Hamiltonian form of Jacobi equation for abnormal geodesics. Considering the defined by (5.7) positive definite quadratic form $\gamma_\tau(u,u)$ on $V^\perp_{\hat{u}(\tau)}$, let us put $\bar{\gamma}_\tau$ for the nonsingular selfadjoint operator $\bar{\gamma}_\tau : V^\perp_{\hat{u}(\tau)} \longrightarrow V^{\perp^*}_{\hat{u}(\tau)}$, which corresponds to γ_τ:

$\gamma_\tau(u,v) = \langle\bar\gamma_\tau u, v\rangle,\ \forall u,v \in V^{\perp}_{\hat u(\tau)}$. Taking an inverse operator $\bar\gamma_\tau^{-1} : V^{\perp^*}_{\hat u(\tau)} \longrightarrow V^{\perp}_{\hat u(\tau)}$, we define a bilinear form γ_τ^{-1} on $V^{\perp^*}_{\hat u(\tau)}$ as $\gamma_\tau^{-1}(u^*,v^*) = \langle\bar\gamma_\tau^{-1}u^*, v^*\rangle,\ \forall u^*,v^* \in V^{\perp^*}_{\hat u(\tau)}$. Obviously for any $x \in \Sigma$ the mapping $u \longrightarrow \sigma(\dot\Upsilon_\tau\cdot, x)$ defines a linear form on $V^{\perp}_{\hat u(\tau)}$, i.e. an element of $V^{\perp^*}_{\hat u(\tau)}$, which depends *linearly* on $x \in \Sigma$. This means, that the correspondence

$$x \longrightarrow \frac{1}{2}\gamma_\tau^{-1}(\sigma(\dot\Upsilon_\tau\cdot, x))$$

defines a quadratic form on Σ.

Treating this quadratic form as time-dependent Hamiltonian on Σ, one may consider on Σ linear Hamiltonian system:

$$\dot x = \dot\Upsilon_\tau\bar\gamma_\tau^{-1}(\sigma(\dot\Upsilon_\tau\cdot, x)), \tag{5.10}$$

which we call *Jacobi equation* for abnormal geodesic $(\hat u(\cdot), \hat q(\cdot), \hat\psi(\cdot), T)$.

If for any $\tau \in [0,T]$ the vectors $u_1(\tau), \dots u_{r-1}(\tau)$ form such basises in $V^{\perp}_{\hat u(\tau)}$, that $\gamma_\tau(u_i(\tau), u_j(\tau)) = \delta_{ij},\ (i,j = 1,\dots r-1)$, then the equation can be presented as

$$\dot x = \sum_{i=1}^{r-1}\sigma(\dot\Upsilon_\tau u_i(\tau), x)\dot\Upsilon_\tau u_i(\tau).$$

Since a Hamiltonian flow preserves symplectic structure of Σ, then the Jacobi equation transforms Lagrangian planes into Lagrangian ones. Therefore one may consider the Hamiltonian flow as a flow on Lagrangian Grassmanian $\mathcal{L}(\Sigma)$. It is generated by the following time-dependent Hamiltonian system on $\mathcal{L}(\Sigma)$:

$$\dot\Lambda = \frac{1}{2}\gamma_\tau^{-1}(\sigma(\dot\Upsilon_\tau\cdot, x))|_\Lambda \tag{5.11}$$

(see Section 2 for details).

Definition 5.1 *Jacobi curve $\tau \to \Lambda_\tau$ $(\tau \in [0,T])$ corresponding to the abnormal geodesic $(\hat u(\cdot), \hat q(\cdot), \hat\psi(\cdot), T)$ is the curve in Lagrangian Grassmanian $\mathcal{L}(\Sigma)$, which starts at $\Lambda_0 = \Pi$, coincides for $\tau \in [0,T)$ with the starting at Π trajectory of the Jacobi equation (5.11) and jumps at $\tau = T-0$ to $\Lambda_T = \Lambda^\Gamma_{T-0} = \Lambda_{T-0}\cap\Gamma^\flat + \Gamma$.* □

Basing on this definition we set

Theorem 5.1 (Index Theorem for Abnormal Geodesics) *Let $\tau \to \Lambda_\tau$ be the Jacobi curve in Lagrangian Grassmanian $\mathcal{L}(\Sigma)$, which corresponds to abnormal geodesic $(\hat u(\cdot), \hat q(\cdot), \hat\psi(\cdot), T)$. Then for any subdivision $\eta_{s+1} = 0 = \eta_0 < \eta_1 < \cdots < \eta_s = T$ of $\tau \to \Lambda_\tau$ into simple subarcs $\Lambda|_{[\eta_i,\eta_{i+1}]},\ (i = 0,\dots s-1)$ Morse index of the abnormal geodesic is equal to*

$$\sum_{i=0}^{s}\mathrm{ind}_\Pi(\Lambda_{\eta_i}, \Lambda_{\eta_{i+1}}) - (n-k), \tag{5.12}$$

where k is corank of the abnormal geodesic path $(\hat u(\cdot), \hat q(\cdot), T)$. □

Proof. We give sketch of the proof of this Theorem, referring for details to the performed in [7] computation of Morse index of an abnormal extremal for Lagrange problem of Calculus of Variations.

Putting $\delta\theta = 0$ in (5.4)-(5.5), we obtain a quadratic form which we call *reduced second variation.* This quadratic form is a Hessian of *input/state mapping* (see [6]) $u(\cdot) \to F(T, u(\cdot))$. Its domain has codimension 1 or 0 in the domain of the second variation (5.4)-(5.5), hence its index is not larger and differs at most by 1 from the index of the second variation. It can be represented as

$$2F''_\varrho|_{[T,\hat u(\cdot),\psi_T]}(0, u(\cdot)) = \int_0^T \sigma\left(\int_0^\tau \Upsilon_\xi u(\xi)d\xi, \Upsilon_\tau u(\tau)\right)d\tau, \tag{5.13}$$

with the domain

$$\{(0,u(\cdot))| \int_0^T \Upsilon_\tau u(\tau)d\tau \in \Pi\}. \tag{5.14}$$

We define Jacobi curve for the reduced 2nd variation following [1].

Definition 5.2 *Jacobi curve $\tau \to \Lambda_\tau^\varrho$ ($\tau \in [0,T]$) corresponding to the reduced second variation (5.13)-(5.14) is the curve in the Lagrangian Grassmanian $\mathcal{L}(\Sigma)$, which starts at $\Lambda_0^\varrho = \Pi$ coincides for $\tau \in [0,T)$ with the starting at Π trajectory of the Jacobi equation (5.11) and jumps at $\tau = T-0$ to $\Lambda_T^\varrho = (\Lambda_T^\varrho)^{\Gamma_\varrho} = \Lambda_T^\varrho \cap \Gamma_\varrho^\flat + \Gamma_\varrho$.* □

The following Proposition providing formula for the index of the reduced second variation (5.13)-(5.14) via Maslov indices of the Jacobi curve Λ_τ^ϱ is corollary of the Theorem 1 in [1].

Proposition 5.2 (Index of the Reduced Second Variation). *Let $\tau \to \Lambda_\tau^\varrho$ $(0 \leq \tau \leq T)$, be Jacobi curve corresponding to the reduced second variation (5.13)-(5.14) along an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. Then for any subdivision $\zeta_{m+1} = 0 = \zeta_0 < \zeta_1 < \cdots < \zeta_m = T$ of $\tau \longrightarrow \Lambda_\tau^\varrho$ into simple subarcs $\Lambda^\varrho|_{[\zeta_i,\zeta_{i+1}]}$, $(i = 0, \ldots m-1)$ the negative index of the reduced second variation (5.13)-(5.14) is equal to*

$$\sum_{i=0}^{m} \mathrm{ind}_\Pi(\Lambda_{\zeta_i}^\varrho, \Lambda_{\zeta_{i+1}}^\varrho) + \dim \cap_{\tau\in[0,T]} \Lambda_\tau^\varrho - (n-k), \tag{5.15}$$

where k is corank of the abnormal geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$. □

Starting from the formula (5.15) one is able to compute negative index of the second variation (5.4)-(5.5), by using following technical Lemma (see [1]).

Proposition 5.3 *Assume, that a quadratic form $Q(\cdot,\cdot)$ is defined on a Hilbert space and is positive definite on a subspace of finite codimension. Let $\mathcal{N}$ be a closed subspace of the Hilbert space, $Q_\mathcal{N}$ be the restriction of Q on $\mathcal{N}$, and $\mathcal{N}_Q^\perp$ be the Q-orthogonal complement to $\mathcal{N}$: $\mathcal{N}_Q^\perp = \{y|\mathcal{B}(x,y) = 0, \forall x \in \mathcal{N}\}$, where $\mathcal{B}$ is corresponding to Q symmetric bilinear form. Then*

$$\mathrm{ind}\, Q = \mathrm{ind}\, Q_\mathcal{N} + \mathrm{ind}\, Q|_{\mathcal{N}_Q^\perp} + \dim(\mathcal{N} \cap \mathcal{N}_Q^\perp) - \dim(\mathcal{N} \cap \ker Q). \square \tag{5.16}$$

To apply the result in our case we take for the Hilbert space $\mathcal{H}$ set of the pairs $(\delta\theta, u(\cdot))$, which meet the condition (5.5), for the subspace $\mathcal{N}$ the set of pairs $(0, u(\cdot))$, which meet the condition (5.14), and for Q the quadratic form (5.4). Evidently $\mathrm{codim}\mathcal{N} \leq 1$.

Following this line the authors have already computed in [7] index of abnormal extremal for Lagrange problem of Calculus of Variations. We have established in [7], that appearance of additional term $\hat{g}\delta\theta$ in the second variation (5.4)-(5.5) in comparison with the reduced second variation (5.13)- (5.14) leads to a change of the final 'jump' of the Jacobi curve, which becomes $\Lambda_T = (\Lambda_{T-0}^{\Gamma_\varrho})^{\hat{g}} = \Lambda_{T-0}^{\Gamma_\varrho} \cap \hat{g}^\flat + \mathrm{span}\{\hat{g}\}$. Since $\hat{g} \in \Gamma_\varrho^\flat$, then $\Gamma_\varrho^{\hat{g}} = \Gamma_\varrho + \mathrm{span}\{\hat{g}\} = \Gamma$ and also $(\Lambda_{T-0}^{\Gamma_\varrho})^{\hat{g}} == \Lambda_{T-0}^\Gamma$. Therefore we come to the formula (5.12) **completing the proof of the Theorem 5.1.** ■.

Now we set Nullity Theorem for abnormal geodesics. Its proof is similar to the given in [7] proof of Nullity Theorem for abnormal extremals of Lagrange problem.

Theorem 5.4 (Nullity Theorem for Abnormal Geoesics) *Let $\tau \longrightarrow \Lambda_\tau$ be the Jacobi curve in Lagrangian Grassmanian $\mathcal{L}(\Sigma)$, which corresponds to a corank k abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$. Then nullity of the abnormal geodesic, i.e. the dimension of the kernel of the second variation (5.4)-(5.5), is equal to* $\dim(\Lambda_T \cap \Pi)$. □

What follows is corollary of the Theorems 5.1 and 5.4 (compare with the Corollary 5.5 in [7]).

Theorem 5.5 (Local Rigidity Condition for Abnormal Geodesics) *Let an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ meet Goh condition (4.11) and Strong Generalized Legendre Condition (4.16). Then for any small enough $\bar{t} > 0$ the restrictions $(\hat{u}(\cdot)|_{[0,\bar{t}]}, \hat{q}(\cdot)|_{[0,\bar{t}]}, \bar{t})$ of the geodesic path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ on $[0, \bar{t}]$ are rigid.* □

Proof. Let us note firstly, that corank of the restrictions $(\hat{u}(\cdot)|_{[0,\bar{t}]}, \hat{q}(\cdot)|_{[0,\bar{t}]}, \bar{t})$ is integer-valued non-decreasing function of t, and hence for small enough $t > 0$ all the restrictions have the same corank $k > 0$.

We are going to prove, that both index and nullity along any restriction $(\hat{u}(\cdot)|_{[0,t]}, \hat{q}(\cdot)|_{[0,t]}, \hat{\psi}(\cdot)|_{[0,t]}, t)$ vanish and then apply the Corollary 4.9.

To compute the index of the restriction $(\hat{u}(\cdot)|_{[0,t]}, \hat{q}(\cdot)|_{[0,t]}, \hat{\psi}(\cdot)|_{[0,t]}, t)$ let us consider corresponding nondecreasing Jacobi curve $\Lambda_\tau|_{[0,t]}$, in Lagrangian Grassmanian. Since $\Gamma \cap \Pi = 0$ then $\Gamma \cap \Lambda_\tau = 0$ for any small enough $\tau > 0$ and therefore $\dim(\Lambda_\tau^\Gamma \cap \Lambda_\tau) = \text{const}$ for small $\tau > 0$. Then according to the Lemma 2.2 there exist $t > 0$ and a Lagrangian plane Δ such that for any $\tau \in [0, t]$ Λ_τ can be connected with Λ_τ^Γ by a simple nondecreasing curve $\Lambda_\tau(s),\ 0 \leq s \leq 1$ such that $\Lambda_\tau(s) \cap \Delta = 0,\ \forall s \in [0, 1]$. Then the concatenation of the curve $\Lambda|_{[0,t]}$ with the corresponding curve $\Lambda_t(s)$ is also simple and evidently nondecreasing. According to the Proposition 5.2 and Theorem 5.1 index of the (having corank 1) restriction $(\hat{u}(\cdot)|_{[0,t]}, \hat{q}(\cdot)|_{[0,t]}, \hat{\psi}(\cdot)|_{[0,t]}, t)$ equals

$$\mathrm{ind}_\Pi(\Pi, \Lambda_t) + \mathrm{ind}_\Pi(\Lambda_t, \Lambda_t^\Gamma) + \mathrm{ind}_\Pi(\Lambda_t^\Gamma, \Pi) - (n-1),$$

where $\Lambda_t^\Gamma = \Lambda_t \cap \Gamma^\flat + \Gamma$. According to the Lemma 2.1 $\mathrm{ind}_\Pi(\Pi, \Lambda_t) + \mathrm{ind}_\Pi(\Lambda_t, \Lambda_t^\Gamma) = \mathrm{ind}_\Pi(\Pi, \Lambda_t^\Gamma)$ for all small enough $t > 0$ and we obtain for the Morse index the expression:

$$\begin{aligned} &\mathrm{ind}_\Pi(\Pi, \Lambda_t^\Gamma) + \mathrm{ind}_\Pi(\Lambda_t^\Gamma, \Pi) - (n-1) = \\ &= 2\frac{1}{2}(n - 1 - \dim(\Lambda_t^\Gamma \cap \Pi)) - (n-1) = -\dim(\Lambda_t^\Gamma \cap \Pi) \leq 0. \end{aligned}$$

Being nonnegative this Morse index must vanish. That implies also $\dim(\Lambda_t^\Gamma \cap \Pi) = 0$, i.e. in virtue of the Theorem 5.4 nullity of the restriction $(\hat{u}(\cdot)|_{[0,t]}, \hat{q}(\cdot)|_{[0,t]}, \hat{\psi}(\cdot)|_{[0,t]}, t)$ also vanishes. ■

6 Distributions Exhibiting Rigidity Phenomenon

We are going to describe some class of distributions, for which rigid abnormal geodesic paths do exist. We will consider germs of smooth $r-$dimensional distributions in R^n. It turns out that some conditions on growth vectors of the distributions provide existence of rigid geodesic paths.

Theorem 6.1 *Let $n \geq 2r,\ q^0 \in M$. Then in the set of 2-jets at q^0 of distributions $\mathcal{D}$ satisfying the condition*

$$\dim \mathcal{D}_q^2 \equiv 2r - 1 \tag{6.1}$$

there is an open subset, such that for any distribution $\mathcal{D}$ satisfying the condition (6.1) with 2-jet lying in this subset, there exists a rigid admissible path starting at q^0. □

Generic 2-dimensional distributions on $n-$dimensional manifold M with $n \geq 4$ not only meet the conditions of the Theorem 6.1, but possess stronger property.

Theorem 6.2 *For any germ at a point $q^0 \in M$ of 2-dimensional distribution $\mathcal{D}$, such that $\mathcal{D}_{q^0}^3 \neq \mathcal{D}_{q^0}^2$, there exists rigid admissible path starting at q^0.* □

Proof of the Theorem 6.1. Let us assume that a distribution $\mathcal{D}$ meeting the conditions of the Theorem 6.1 is spanned by the vector fields $g^1, \ldots g^r$, while

$$\mathcal{D}^2 = \mathrm{span}\{g^1, \ldots g^r, [g^1, g^2], \ldots [g^1, g^r]\}. \tag{6.2}$$

Assume that for some $\psi^0 \in \mathcal{T}^*_{q^0}M \setminus 0$, annihilating $\mathcal{D}^2$ the $((r-1)\times r)-$matrix

$$A_\psi = \begin{pmatrix} \psi[g^1,[g^1,g^2]] & & \psi[g^r,[g^1,g^2]] \\ \vdots & \cdots & \vdots \\ \psi[g^1,[g^1,g^r]] & & \psi[g^r,[g^1,g^r]] \end{pmatrix},$$

has the maximal rank $(r-1)$. Then it holds also for a nonzero ψ close to ψ^0 and there exists a smoothly depending on ψ solution $u(\psi) = (u_1(\psi), \ldots u_r(\psi)) \in R^r \setminus 0$ of the systems $A_\psi u(\psi) = 0$; Without loss of generality we may assume $u(\psi^0) = (1, 0 \ldots 0)$.

Assume in addition that the quadratic form

$$L^{\psi^0}(v_2, \ldots, v_r) = \psi^0[[Gu(\psi^0), Gv], Gv](q^0) = \sum_{i,j=2}^{r} \psi^0[[g^1,g^i],g^j](q^0)v_iv_j$$

is positive definite. Hence for ψ close to ψ^0 in $\mathcal{T}^*M$ the quadratic forms

$$L^{\psi}(v) = \psi[[Gu(\psi), Gv], Gv](q) \tag{6.3}$$

are also positive definite on the orthogonal complements V_ψ to $u(\psi)$ in R^r.

Any distribution meeting the conditions of the theorem and with 2-jet belonging to a small enough neighborhood of the 2-jet of $\mathcal{D}$ meets the above mentioned assumptions as well.

For any such distribution let us introduce a Hamiltonian $h(q,\psi) = \sum_{i=1}^r u_i(\psi)\psi g^i(q)$. We shall demonstrate that some subarc of the starting at (q^0,ψ^0) trajectory $(\hat{q}(\cdot), \hat{\psi}(\cdot))$ of the corresponding Hamiltonian system is an abnormal geodesic, which meets Goh and Strong Generalized Legendre Conditions.

The Strong Generalized Legendre Condition along a small subarc starting at (q^0,ψ^0) is implied by the positive definiteness of the quadratic forms (6.3). We have to verify, that $\psi \perp \mathcal{D}^2$ along the trajectory. In virtue of (6.2) it is enough to establish the equalities

$$\psi g^i \equiv \psi[g^1,g^i] \equiv 0, \ i = 1, \ldots, r.$$

along the trajectory.

Let us put $h_i(t) = \psi g^i(\hat{q}(t), \hat{\psi}(t)), h_{1i}(t) = \psi[g^1,g^i](\hat{q}(t), \hat{\psi}(t)), \ (i = 1, \ldots, r)$ and compute $\dot{h}_i, \ \dot{h}_{1i}$.

Since $h = \sum_{j=1}^r u_j(\psi)h_j$, then

$$\dot{h}_i = \{h, h_i\}(\hat{q}(t), \hat{\psi}(t)) = \{\sum_{j=1}^{r} u_jh_j, h_i\}(\hat{q}(t), \hat{\psi}(t)) =$$
$$= \sum_{j=1}^{r} u_j(\psi)\{h_j, h_i\}(\hat{q}(t), \hat{\psi}(t)) + \sum_{j=1}^{r}\{u_j, h_i\}h_j(\hat{q}(t), \hat{\psi}(t)).$$

In virtue of (6.2) $[g^j, g^i]$ lies in the linear span of vector fields $g^1, \ldots g^r, [g^1,g^2], \ldots [g^1,g^r]$; hence

$$\{h_j, h_i\} = \psi[g^i, g^j] = \sum_{k=2}^{r} \alpha_k(x)\psi[g^1,g^k] = \sum_{k=2}^{r} \alpha_k h_{1k},$$

and therefore

$$\dot{h}_i(\hat{q}(t), \hat{\psi}(t)) = \sum_{j=1}^{r}\{u_j, h_i\}h_j(\hat{q}(t), \hat{\psi}(t)) + \sum_{j=1}^{r} v_j(\psi)h_{1j}(\hat{q}(t), \hat{\psi}(t)). \tag{6.4}$$

Also

$$\dot{h}_{1i}(\hat{q}(t), \hat{\psi}(t)) = \{h, h_{1i}\}(\hat{q}(t), \hat{\psi}(t)) = \{\sum_{j=1}^{r} u_jh_j, h_{1i}\}(\hat{q}(t), \hat{\psi}(t)) =$$

$$= \sum_{j=1}^{r}(u_j\psi[g^j,[g^1,g^i]](\hat{q}(t),\hat{\psi}(t)) + \{u_j,h_{1i}\}h_i)(\hat{q}(t),\hat{\psi}(t)) =$$

$$= A_\psi u(\psi)(\hat{q}(t),\hat{\psi}(t)) + \sum_{j=1}^{r}\{u_j,h_{1i}\}h_i(\hat{q}(t),\hat{\psi}(t)) = \sum_{j=1}^{r}\{u_j,h_{1i}\}h_i(\hat{q}(t),\hat{\psi}(t)). \qquad (6.5)$$

Therefore the functions $h_i(\hat{q}(t),\hat{\psi}(t)),\ h_{1i}(\hat{q}(t),\hat{\psi}(t)),\ (i=1,\dots r)$ satisfy the linear system of differential equations (6.4)-(6.5), and since

$$h_i(q^0,\psi^0) = \psi^0 g^i(q^0) = h_{1i}(q^0,\psi^0) = \psi^0[g^1,g^i](q^0) = 0,\ i=1,\dots r,$$

then

$$h_i(\hat{q}(t),\hat{\psi}(t)) = h_{1i}(\hat{q}(t),\hat{\psi}(t)) \equiv 0\ (i=1,\dots,r).$$

We have established that the abnormal geodesic $(\hat{q}(\cdot),\hat{\psi}(\cdot))$ satisfies the conditions of the Theorem 5.5 and therefore is rigid.

To finish the proof of the Theorem 6.1 we only have to construct at least one $r-$dimensional distribution $\mathcal{D}$, with the growth vector and the basis meeting all the assumptions we have done.

Let $q^0 = 0_{R^n}$ and $R^n = \{(x,y,z)|x \in R^r,\ y \in R^r,\ z \in R^{n-2r}\}$. Let us put

$$g^1 = \partial/\partial x_1;\ g^i = \partial/\partial x_i + x_1\partial/\partial y_i + y_i\partial/\partial y_1,\ i=2,\dots,r-1;$$

$$g^r = \partial/\partial x^r + x^1\partial/\partial y_r + y_r\partial/\partial y_1 + \sum_{j=1}^{n-2r} x_1^{j+1}\partial/\partial z_j.$$

The vector fields $g^1,\dots,g^r$ span germ of $r-$dimensional distribution of full Lie rank. In addition:

$$[g^1,g^i] = \partial/\partial y_i,\ i=2,\dots,r-1;\ [g^1,g^r] = \partial/\partial y_r + \sum_{j=1}^{n-2r}(j+1)x_1^j\partial/\partial z_j;$$

$$[g^i,g^j] = 0,\ i,j=2,\dots r;\ [g^1,[g^1,g^i]] = 0,\ i=2,\dots,r-1;$$

$[g^1,[g^1,g^r]]$ is linear combination of $\partial/\partial z_j$, and $[[g^1,g^i],g^j] = \delta_{ij}\partial/\partial y_1$, where δ_{ij} is Kronecker symbol. All the above mentioned assumptions will hold if we choose ψ^0 such that

$$\psi^0\partial/\partial x_i = \psi^0\partial/\partial z_j = 0,\ i,j=1,\dots,r;\ \psi^0\partial/\partial y_i = 0,\ i=2,\dots,r;\ \psi^0\partial/\partial y_1 = 1. \blacksquare$$

Proof of the Theorem 6.2. One can always choose vector fields f,g, which span $\mathcal{D}$ and a covector $\psi^0 \in \mathcal{T}^*_{q^0}M \setminus 0$, in such a way, that following conditions hold:

$$\psi^0 f(q^0) = \psi^0 g(q^0) = \psi^0[f,g](q^0) = \psi^0[f,[f,g]](q^0) = 0,$$

$$\psi^0[g,[f,g]](q^0) > 0 \qquad (6.6)$$

(it is enough to chose $\psi^0 \perp \mathcal{D}^2$, $\psi^0 \not\perp \mathcal{D}^3$ and, when necessary, multiply ψ^0 by -1). The inequality (6.6) holds for all ψ close to ψ^0 in $\mathcal{T}^*M$. Considering the equation

$$\Phi(\psi,u) = \psi[f+gu,[f,g]](q) = 0$$

we note, that, since $\psi^0[g,[f,g]](q^0) > 0$ then locally the equation $\Phi(\psi,u)=0$ has smooth solution $u(\psi) = (\psi[f,[f,g]](q))/(\psi[g,[f,g]](q));\ u(\psi^0)=0$. As in the proof of the previous theorem some subarc of starting at (q^0,ψ^0) trajectory of Hamiltonian system with the Hamiltonian $h = \psi(f+gu(\psi))$ is abnormal geodesic, which meets Goh and Strong Generalized Legendre Condition and hence is rigid. $\blacksquare$

7 Abnormal Geodesics for 2-dimensional Distributions: Few More Steps

In this Section we deal with abnormal geodesics for $2-$*dimensional* smooth distributions. For this case we are able to proceed further with the computation of Morse index and nullity and obtain elegant representation of the Jacobi equation and characterization of conjugate points.

Let us consider a $2-$dimensional distribution $\mathcal{D}$ on a $(n+2)-$dimensional manifold; let the vector fields $f, g \in \mathrm{Vect}M$ span $\mathcal{D}$. Assume that:

i) the vector fields

$$f, g, [f,g], \ldots (\mathrm{ad} f)^{n-1} g$$

are linearly independent at every point of the domain we treat;

ii) $(\mathrm{ad} f)^n g$ can be presented as a linear combination with $C^\infty-$coefficients of these $n+1$ vector fields:

$$(\mathrm{ad} f)^n g = \beta f + \sum_{i=0}^{n-1} \alpha^i (\mathrm{ad} f)^i g \ (\beta, \alpha^i \in C^\infty(M)). \tag{7.1}$$

Then the trajectories of the vector field f are corank 1 abnormal geodesics for the distribution $\mathcal{D}$. Let us consider distribution (free $C^\infty(M)-$module of vector fields)

$$V = \mathrm{span}\{f, g, [f,g], \ldots (\mathrm{ad} f)^{n-1} g\},$$

and assume, that:

iii) in the treated domain

$$[[f,g]g]](q) \notin V(q).$$

Let ψ be a $1-$form on the domain, defined by the conditions:

$$\psi \perp V, \ \psi[[f,g]g] = 1.$$

We shall derive Jacobi equation for the abnormal geodesic, which corresponds to the vector field f. We denote by $\hat{q}(\cdot) = q^0 \circ e^{tf}$ the starting at $q^0 = \hat{q}(0)$ trajectory of f; $\hat{q}(T) = q^1$. Following the approach of the Section 5 let us consider skewsymmetric bilinear form $(v_1, v_2) \to \psi \cdot [v_1, v_2](q^1), \ v_1, v_2 \in V$. Taking quotient of V w.r.t. the kernel of this form we obtain a $2(n+1)-$dimensional symplectic space Σ'. We reduce the symplectic space considering the $(2n+1)-$dimensional skeworthogonal complement to the canonical projection $\hat{g}$ of the vector field f onto Σ' and then taking quotient of Σ' w.r.t. $\mathrm{span}\{\hat{g}\}$. The result is denoted further by Σ; it is $2n-$dimensional symplectic space with skewscalar product denoted by σ. We again denote by $\underline{Y}$ the image of a vector field $Y \in V$ under the canonical projection $V \to \Sigma$.

We are going to introduce special coordinates in Σ and to derive one more representation of the Jacobi equation (5.10).

Let us put for $i \geq 0$

$$g_t = e^{(t-T)\mathrm{ad} f} g, \ g_t^i = e^{(t-T)\mathrm{ad} f} (\mathrm{ad} f)^i g = \partial^i g_t / \partial t^i,$$

$$\gamma_t^i = \hat{\psi} \cdot [g_t^1, g_t^i](q^1) = \hat{\psi}(t)[[f,g], (\mathrm{ad} f)^i g](\hat{q}(t)).$$

Returning to the equality (7.1) we put $\alpha_t^i = \alpha^i(\hat{q}(t)) \ (i = 0, \ldots n-1), \ \beta_t = \beta(\hat{q}(t))$, and derive from (7.1):

$$g_t^n(q^1) = \beta_t f(q^1) + \sum_{i=0}^{n-1} \alpha_t^i g_t^i(q^1).$$

Lemma 7.1

$$\underline{g}_t^n = \sum_{i=0}^{n-1} \alpha_t^i \underline{g}_t^i. \ \square \tag{7.2}$$

Proof. Chosing coordinates in Σ as in (5.3) (with $k=1$) we only need to establish, that

$$\partial(\psi \cdot g_t^n)/\partial x|_{q^1} = \sum_{i=0}^{n-1} \alpha_t^i \partial(\psi \cdot g_t^i)/\partial x|_{q^1} + \beta_t \partial(\psi \cdot f)/\partial x|_{q^1}$$

for local coordinates $x=(x_1,\ldots x_n)$ in the neighborhood of $q^1 \in M$. But this follows directly from (7.1) and the equalities $(\psi g_t^i)|_{q^1} \equiv 0,\ i=0,\ldots n-1$. ■

Let Π be image under the canonical projection of the vector fields Y, which meet the condition $\psi \cdot [f,Y](q^1)=0$ and vanish at q^1; Π is Lagrangian plane in Σ. It follows from (7.1)-(7.2), that $\Sigma = \Pi \oplus \mathrm{span}\{\underline{g}_t, t \in R\}$ and for any $\tau \in R$ the vectors $\underline{g}_\tau, \underline{g}_\tau^1 \ldots \underline{g}_\tau^{n-1}$ form the basis of the subspace $\mathrm{span}\{\underline{g}_t, t \in R\} = \Delta$. Let us emphasize, that the subspace Δ *is not* Lagrangian and σ defines a nondegenerate coupling between Π and Δ.

Representing a $x \in \Sigma$ as $x = z + \xi$, where $z \in \Delta,\ \xi \in \Pi$, we may write the Jacobi equation (see (5.10)) in these coordinates as

$$\gamma_t^0(\dot{z}+\dot{\xi}) = \sigma(\underline{g}_t^1, z+\xi)\underline{g}_t^1,$$

or

$$\gamma_t^0 \dot{z} = \sigma(\underline{g}_t^1, z)\underline{g}_t^1 + \sigma(\underline{g}_t^1, \xi)\underline{g}_t^1,\ \dot{\xi} = 0. \tag{7.3}$$

Evidently one of the solutions of this equation is: $z_t \equiv \underline{g}_t, \xi_t = 0$.

We call $\bar{t}$ *a conjugate point of multiplicity* $k>0$, *for the abnormal geodesic* $\hat{q}(t) = q^0 \circ e^{tf}$, if for the equation (7.3) the space of solutions, which satisfy boundary conditions

$$z_0 = 0,\ z_{\bar{t}} \parallel \underline{g}_{\bar{t}},\ \sigma(\underline{g}_0, \xi_0) = 0, \tag{7.4}$$

is $k-$dimensional.

Let us put $\zeta_t = \sigma(\underline{g}_t, \xi_0)$ and present z_t in the form: $z_t = \sum_{i=0}^{n-1} z_t^i \underline{g}_t^i$. Then the equation (7.3) can be transformed into the following system

$$\begin{array}{c}\gamma_t^0(\dot{z}^1 + \alpha_t^1 z^{n-1}) = \sum_{j=2}^{n-1} \gamma_t^j z^j + \dot{\zeta},\ z_0^1 = 0 \\ \dot{z}^j + \alpha_t^j z^{n-1} = -z^{j-1},\ z_0^j = 0,\ j=2,\ldots n-1, \\ \zeta^{(n)} = \sum_{i=0}^{n-1} \alpha_t^i \zeta^{(i)},\ \zeta_0 = 0.\end{array} \tag{7.5}$$

(the equation for z^0, which enters neither (7.4) nor (7.5), is ommitted):

The multiplicity of a conjugate point is dimension of the space of those solutions of the system (7.5), which satisfy the conditions

$$z_{\bar{t}}^i = 0,\ i=1,\ldots n-1. \tag{7.6}$$

Summarizing the aforesaid we set following

Theorem 7.1 *Assume that for starting at q^0 trajectory $\hat{q}(t) = q^0 \circ e^{tf}$ of 2-dimensional distribution on an $(n+2)-$dimensional manifold the set above conditions i),ii) and iii) hold. Then:*

1) *$\hat{q}(t),\ t \in [0,T]$, is corank 1 abnormal geodesic path of the distribution;*

2) *it has finite number (may be zero) of conjugate points $\bar{t}_i$ and multiplicity of a conjugate point $\bar{t}$ is equal to the dimension of the space of solutions of the system (7.5), which satisfy the boundary conditions (7.6);*

3) *Morse index of the abnormal geodesic is equal to the sum of multiplicities of the conjugate points, which are located on $(0,T)$;*

4) *nullity of the abnormal geodesic is equal to the multiplicity of conjugate point at T (vanishes if T is not a conjugate point);*

5) *for the abnormal geodesic path to be rigid it is necessary, that $(0,T)$ does not contain conjugate points, and it is sufficient, that $(0,T]$ does not contain conjugate points.* □

Proof. Statement 1) was established at the beginning of the Section, finiteness of the set of conjugate points and the statements 3),4) follow from *strong regularity* (see [21]) of the abnormal geodesic $\hat{q}(\cdot)$; statement 5) follows from corank 1 variants of the Theorems 4.1 and 4.8 together with 3) and 4). ■

Now we treat in more detailed way the case $n = 2$, i.e. 2-dimensional distributions on 4-dimensional manifolds. Here the vector field f, which meets the condition (7.1) exists and is unique for any 2-dimensional distribution of maximal growth; such distributions define so-called *Engel structure* on 4-dimensional manifolds (readers can find in [11] a detailed survey of different problems connected with these structures). For $n = 2$ the system (7.5) takes form:

$$\dot{z}^1 = -\alpha_t^1 z^1 + \frac{\dot{\zeta}}{\gamma_t^0}, \; z_0^1 = 0,$$

$$\ddot{\zeta} = \alpha_t^0 \zeta + \alpha_t^1 \dot{\zeta}, \; \zeta_0 = 0, \dot{\zeta}_0 = 1. \tag{7.7}$$

Besides

$$\dot{\gamma}_t^0 = d(\psi[g_t^1, g_t])/dt = \psi[g_t^2, g_t] = \alpha_t^1 \gamma_t^0, \; \gamma_0^0 = 1.$$

Hence $\gamma_t^0 = e^{\int_0^t \alpha_\tau^1 d\tau}$, and therefore

$$z_t^1 = \int_0^t \frac{1}{\gamma_\tau^0} e^{-\int_\tau^t \alpha_\theta^1 d\theta} \dot{\zeta}_\tau d\tau = \zeta_t e^{-\int_0^t \alpha_\tau^1 d\tau}.$$

One sees, that $\bar{t}$ is a conjugate point if and only if $\zeta_{\bar{t}} = 0$. Multiplicity of any conjugate point equals to 1.

Therefore for the 2-dimensional case the following corollary of the Theorem 7.1 is valid.

Corollary 7.2 *For every 2-dimensional distribution of maximal growth on 4-dimensional manifold M and for every point $q^0 \in M$ there exists a corank 1 abnormal geodesic path $\hat{q}(t)$, $(t \in [0,T])$ of the distribution starting at q^0. Morse index of the corresponding abnormal geodesic is equal to the number of located on $(0,T)$ zeros of the solution $\zeta(\cdot)$ of the equation (7.7). For the abnormal geodesic path to be rigid it is necessary (correspondingly, sufficient), that there are no zeros of $\zeta(\cdot)$ on $(0,T)$ (correspondingly, on $(0,T]$).* □

8 Rigid Trajectories of Affine Control Systems

In the Section we extend our approach onto the class of affine control systems

$$\dot{q} = f(q) + \sum_{i=1}^{r} g^i(q) u_i, \; q(0) = q^0 \tag{8.1}$$

and derive rigidity conditions for extremals of these systems.

Here the *drift vector field* $f(q)$ and the *control vector fields* $g^i(q)$, $i = 1, \ldots r$, are C^∞ on M. Admissible controls $u(\cdot) = (u_1(\cdot), \ldots u_r(\cdot)) \in L_\infty^r$.

The material of this section relates to the results of [7], where nonlinear Lagrange problem of the Calculus of Variations was treated, and also with [10], which treated time-optimal problems for affine control systems.

We start with definition of rigidity for a given input $\hat{u}(t)$, $t \in [0,T]$ of the affine control system (8.1), We assume $\hat{u}(\cdot)$ to be continuous at $T-0$. The extension of $\hat{u}(\cdot)$ from $[0,T]$ onto $[0,T+\delta]$ by the constant $\hat{u}(T)$ will be denoted also by $\hat{u}(\cdot)$. We assume that the starting at q^0 trajectory $\hat{q}(\cdot)$ of the system (8.1) driven by the control $\hat{u}(\cdot)$ exists on $[0,T+\delta]$.

Definition 8.1 *A control $\hat{u}(\cdot)$ and the corresponding trajectory $\hat{q}(\cdot)$ of the control system (8.1) are called rigid on $[0,T]$, if for some $\epsilon > 0$ no one (different from $\hat{u}(\cdot)$) control from $\delta-$neighborhood of $\hat{u}(\cdot)$ in L_∞^r can steer the system (8.1) from q^0 to $\hat{q}(T)$ in a time $T' \in [T-\delta, T+\delta]$.* □

Following the approach of the Section 4 we consider for the system (8.1) a *time$\times$input/state mapping* $F : R \times L^r_\infty \to M$; F maps a pair $(t, u(\cdot))$ consisting of time moment t and an admissible control $u(\cdot)$ into the point $q(t)$ of the trajectory $q(\cdot)$ of the control system (8.1).

It turns out that for $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ to be rigid, $(T, \hat{u}(\cdot))$ has to be critical point of F. That means $\mathrm{Im}F'|_{(T,\hat{u}(\cdot))} \neq \mathcal{T}_{\hat{q}(T)}M$, and there exists nonzero $\hat{\psi}_T \in \mathcal{T}^*_{\hat{q}(T)}M$ annihilating $\mathrm{Im}F'|_{(T,\hat{u}(\cdot))}$. For any such $\hat{\psi}_T$ one can transform the equality $\hat{\psi}_T F' \equiv 0$ into Hamiltonian form of the Euler-Lagrange equation (compare with the Theorem 3.1), which is extremality condition for the path $(\hat{u}(\cdot), \hat{q}(\cdot))$. It is more convenient for us to set it this time as a Definition.

Definition 8.1 (Extremality) *We call $(\hat{u}(\cdot), \hat{q}(\cdot))$ extremal pair for the affine control system (8.1) on $[0, T]$, if there exists an absolutely-continuous covector-function $\hat{\psi}(\tau)$ on $[0, T]$) such that the triple $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot))$:*
1) satisfies Hamiltonian system

$$\dot{q} = \partial H/\partial\psi, q(0) = q^0, \tag{8.2}$$

$$\dot{\psi} = -\partial H/\partial q, \tag{8.3}$$

with an 'affine' Hamiltonian

$$H_f(u, q, \psi) = \psi \cdot (f(q) + G(q)u); \tag{8.4}$$

2)meets stationarity condition

$$\frac{\partial H_f}{\partial u}|_{(\hat{u}(\tau),\hat{q}(\tau),\hat{\psi}(\tau))} = \hat{\psi}(\tau)G(\hat{q}(\tau)) = 0, \text{ for almost all } \tau \in [0, T], \tag{8.5}$$

and 'transversality condition'

$$H_f(\hat{u}(\tau), \hat{q}(\tau), \hat{\psi}(\tau)) = 0, \text{for almost all } \tau \in [0, T]. \ \Box \tag{8.6}$$

We call $\hat{u}(\cdot)$ *extremal control* and $\hat{q}(\cdot)$ *extremal trajectory* of the control system (8.1) on $[0, T]$.

The quadruple $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ is called *extremal* of the control system (8.1) on $[0, T]$.

Corank of an extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is the dimension of the space of extremals, it enters, or, equivalently, dimension of the space of those $\hat{\psi}(\cdot)$, which together with $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ satisfy the Definition (8.1), or, equivalently, $\mathrm{corank}\, F'|_{(T,\hat{u}(\cdot))}$ at $\mathcal{T}_{\hat{q}(T)}M$.

It follows from the Implicit Function Theorem, that extremality is necessary for rigidity.

Proposition 8.1 *A rigid path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ of the control system (8.1) must be extremal pair of the system.* $\Box$

To derive 2nd order rigidity conditions we have to involve again Goh and Generalized Legendre Conditions along extremal. They are (compare with (4.11) and (4.15)):

$$\hat{\psi}(\tau) \cdot [Gv, Gw](\hat{q}(\tau)) = 0, \ \forall v, w \in R^r, \tag{8.7}$$

and for all $\tau \in [0, T]$:

$$\gamma_\tau(v, v) = \hat{\psi}(\tau) \cdot [Gv[\hat{f}_\tau, Gv]](\hat{q}(\tau)) \geq 0, \ \forall v \in R^r. \tag{8.8}$$

Strong Generalized Legendre Condition (compare with (4.16)) looks like follows: for some $\beta > 0$ and for all $\tau \in [0, T]$:

$$\gamma_\tau(v, v) = \hat{\psi}(\tau) \cdot [Gv[\hat{f}_\tau, Gv]](\hat{q}(\tau)) \geq \beta\|v\|^2, \ \forall v \in R^r. \tag{8.9}$$

Following result is 'affine version' of the Proposition 4.3.

Proposition 8.2 (Goh and Legendre Necessary Rigidity Conditions) *For an extremal trajectory of the affine control system (8.1) to be rigid it is necessary, that any abnormal geodesic, it enters, satisfy the Goh condition (8.7) and the Generalized Legendre Condition (8.8).* □

Let us denote by $\bar{\gamma}_\tau$ the nonsingular symmetric $(r \times r)-$matrix, which corresponds to the quadratic form $\gamma_\tau(v,v)$ on R^r. Let $\gamma_{ij}^{-1}(\tau)$ be an $ij-$entry of the matrix $\bar{\gamma}_\tau^{-1}$.

Following proposition is 'affine version' of the Theorem 4.4.

Proposition 8.2 (Regularity of Extremals for Affine Systems) *If Goh condition (8.7) and Strong Generalized Legendre Condition (8.9) hold along an extremal $(\hat{u}(\cdot),\hat{q}(\cdot),\hat{\psi}(\cdot),T)$ of the control system (8.1), then the extremal control $\hat{u}(\tau)$ is smooth and can be calculated as:*

$$\hat{u}_i(\tau) = -\sum_{j=1}^{r} \gamma_{ij}^{-1}(\tau)\hat{\psi}(\tau)\cdot[f,[f,g_j]](\hat{q}(\tau)),\ i = 1,\dots r.\ \square$$

We define correspondingly 1st and 2nd variations of the system (8.1) along its extremal $(\hat{u}(\cdot),\hat{q}(\cdot),\hat{\psi}(\cdot),T)$ as the differential and the Hessian of the time$\times$input/state mapping $F(t,u(\cdot)$ at the point $(T,\hat{u}(\cdot)) \in R \times L_\infty^r$.

The formulae are: for the first variation

$$F'|_{(T,\hat{u}(\cdot))}(\delta\theta,u(\cdot)) = \hat{f}_T(q^1)\delta\theta + \int_0^T Y_\tau(q^1)u(\tau)d\tau, \tag{8.10}$$

and for the second variation

$$2F''|_{(T,\hat{u}(\cdot))}[\hat{\psi}_T](\delta\theta,u(\cdot)) = \hat{\psi}_T\cdot(\int_0^T[-\hat{f}_T\delta\theta + \int_0^\tau Y_\xi u(\xi)d\xi, Y_\tau u(\tau)](q^1)d\tau), \tag{8.11}$$

with the domain

$$\{(\delta\theta,u(\cdot)) \in (R \times L_\infty^r)|\hat{f}_T(q^1)\delta\theta + \int_0^T Y_\tau(q^1)u(\tau)d\tau = 0, \tag{8.12}$$

where

$$\hat{f}_\tau(q) = f(q) + \sum_{i=1}^{r} g^i(q)\hat{u}_i(\tau),\ Y_\tau v = \mathrm{Ad}(\overrightarrow{\exp}\int_T^\tau \hat{f}_\xi d\xi)Gv\ \forall v \in R^r. \tag{8.13}$$

We define *nullity* and *Morse index of extremal* $(\hat{u}(\cdot),\hat{q}(\cdot),\hat{\psi}(\cdot),T)$ as the dimension of the kernel and negative index of the quadratic form (8.11) $-$ (8.12). *Morse index of extremal path* $(\hat{u}(\cdot),\hat{q}(\cdot),T)$ is minimum of the indices of the extremals $(\hat{u}(\cdot),\hat{q}(\cdot),\hat{\psi}(\cdot),T)$, which this path enters.

The following Proposition is direct generalization of the Propositions 4.5 and 4.7.

Proposition 8.3 *If Goh condition (8.7) and Generalized Strong Legendre condition (8.9) hold along extremal $(\hat{u}(\cdot),\hat{q}(\cdot),\hat{\psi}(\cdot),T)$ of the afiine control system (8.1), then:*

1) the second variation (8.11) $-$ (8.12) *can be extended by continuity onto the space $R \times H_{-1}^r[0,T]$ and is weak positive definite on some subspace of finite codimension in $R \times H_{-1}^r[0,T]$, i.e. admits on this subspace a lower estimate:*

$$2F''|_{(T,\hat{u}(\cdot))}[\hat{\psi}_T](\delta\theta,u(\cdot)) \geq c(\|u(\cdot)\|_{-1}^2 + \delta\theta^2); \tag{8.14}$$

2)the kernel of the second variation is contained in $R \times (L_\infty^r \oplus H_{-1}^r[0] \oplus H_{-1}^r[T])$, moreover for an element $(\theta,u(\cdot))$ of the kernel its second component $u(\cdot)$ is C^∞ on $(0,T)$. □

The following two theorems are slight generalizations of the obtained in the Section 4 rigidity conditions for distributions.

Theorem 8.4 (Necessary Rigidity Conditions for Affine Systems) *If a corank k extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ of the system (8.1) is rigid on $[0, T]$, then its Morse index should not exceed $k-1$. In particular for a rigid corank 1 extremal path the index must vanish.* $\square$

Theorem 8.5 (Sufficient Rigidity Conditions for Affine Systems) *1) If the second variation along an extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the afiine control system (8.1) is weak positive definite, then the extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is rigid. 2)In particular, if Goh condition (8.7) and Strong Generalized Legendre condition (8.9) hold along the extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ and Morse index and nullity of the extremal both vanish, then the extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ is rigid.* $\square$

To compute Morse index and nullity which play an important role for the rigidity conditions for extremal paths of affine control system (8.1) we have to repeat almost literally what was done in the Section 5. We refer to that Section marking only minor differences.

Given an extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the affine control system (8.1), we define linear space W (compare with (5.1)) as:

$$W = \mathrm{span}\{\{\hat{f}_T(q^1)\} \cup \{Y_\tau(q^1)v | \tau \in [0, T], v \in R^r\}\},$$

where $\hat{f}_T$ and Y_τ are defined by (8.13). Evidently $W = \mathrm{Im} F'|_{(T, \hat{u}(\cdot))}$ and $\mathrm{codim}\, W = k$ is corank of the extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$.

Introducing like in the Section 5 the symplectic space Σ ($\dim \Sigma = 2(n-k)$), Lagrangian plane Π and denoting by $\underline{f}$ the canonical projection of the vector field $\hat{f}_T$ we see, that (5.4)-(5.5) is as well symplectic representation for the second variation $(8.11) - (8.12)$ of the affine control system (8.1).

Therefore the Jacobi equation for the extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the affine control system (8.1) has the same form (5.10) or (5.11). Introducing isotropic subspace:

$$\Gamma_f = \mathrm{span}\{\underline{f} \cup \underline{Gv} | v \in R^r\} \subset \Sigma,$$

we define Jacobi curve in Lagrangian Grassmanian for the extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ (compare with the Definition 5.1).

Definition 8.2 (Jacobi curve for extremal of affine system) *Jacobi curve corresponding to an extremal of the affine control system (8.1) is a curve $\tau \to \Lambda_\tau (\tau \in [0, T])$ in Lagrangian Grassmanian $\mathcal{L}(\Sigma)$, which coincides for $\tau \in [0, T)$ with the starting at Π trajectory of the Jacobi equation (5.11) in $\mathcal{L}(\Sigma)$ and jumps at $T-0$ to $\Lambda_T = \Lambda_{T-0}^{\Gamma_f} = \Lambda_{T-0} \cap \Gamma_f^\flat + \Gamma_f$.* $\square$

As for abnormal sub-Riemannian geodesics Morse index and nullity of the extremal of affine system can be computed via symplectic invariants of the Jacobi curve.

Theorem 8.6 (Index Theorem for Extremals of Affine System) *Let $\tau \to \Lambda_\tau (\tau \in [0, T])$, be the Jacobi curve, which corresponds to an extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the affine control system (8.1). Then for any subdivision $\eta_{s+1} = 0 = \eta_0 < \eta_1 < \cdots < \eta_s = T$ of $\tau \to \Lambda_\tau$ into simple subarcs $\Lambda|_{[\eta_i, \eta_{i+1}]}$, $(i = 0, \ldots s-1)$ Morse index of the extremal equals to*

$$\sum_{i=0}^{s} \mathrm{ind}_\Pi(\Lambda_{\eta_i}, \Lambda_{\eta_{i+1}}) - (n-k), \tag{8.15}$$

where k is the corank of the extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$. $\square$

Theorem 8.7 (Nullity Theorem for Extremals of Affine System) *Let $\tau \to \Lambda_\tau$ $(\tau \in [0, T])$, be the Jacobi curve, which corresponds to an extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the affine control system (8.1). Then nullity of the extremal, i.e. the dimension of the kernel of the second variation* $(8.11) - (8.12)$, *equals to* $\dim(\Lambda_T \cap \Pi)$. $\square$

Theorem 8.8 (Local Rigidity for Extremals of Affine System) *Let an extremal $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$ of the affine control system (8.1) meet Goh condition (8.7) and Strong Generalized Legendre Condition (8.9). Then for any small enough $\bar{t} > 0$ the restrictions $(\hat{u}(\cdot)|_{[0,\bar{t}]}, \hat{q}(\cdot)|_{[0,\bar{t}]}, \bar{t})$ of the extremal path $(\hat{u}(\cdot), \hat{q}(\cdot), T)$ on $[0, \bar{t}]$ are rigid.* $\square$

9 Appendix: Isolated Points on Critical Levels of Smooth Mappings and Rigidity of Abnormal Geodesics

In the Section 3 we have represented the problem of finding minimal geodesics as Lagrange Problem of Calculus of Variations, which is in turn particular case of problem of relative extremum:

$$\mathcal{J}(x) \longrightarrow \min, \ F(x) = 0, \tag{9.1}$$

where $\mathcal{J}$ is a scalar function(al) on a Banach space X and F maps X into R^k.

Necessary 1st order condition for local minimality of point $\hat{x} \in X$ for this problem is *extremality* of $\hat{x}$. Namely, if $\hat{x}$ is point of minimum and $\mathcal{J}, F$ are Frechet differentiable at $\hat{x}$, then there exists a nonzero pair of Lagrange multipliers $(\lambda_0, \lambda) \in R_+ \times R^{k^*}$, such that $\hat{x}$ is critical point for Lagrangian $\mathcal{L} = \lambda_0 \mathcal{J}(x) + \lambda F(x)$ (Lagrange multipliers rule). We call such $\hat{x}$ *extremal point* and $(\hat{x}, \lambda_0, \lambda)$ *extremal* for the problem (9.1).

Evidently an extremal point $\hat{x}$ may enter different extremals with different Lagrange multipliers; *corank of an extremal point* is the dimension of the space of Lagrange multipliers, which correspond to it.

An extremal $(\hat{x}, \lambda_0, \lambda)$ is *normal*, whenever $\lambda_0 \neq 0$, and *abnormal* otherwise. We use the notation $(\hat{x}, \lambda)$ for abnormal extremals. If $\lambda_0 = 0$, then the functional $\mathcal{J}$ does not at all enter the 1st-order condition. Since we suppose to deal only with abnormal extremals, then we may at all forget about the functional $\mathcal{J}$ and at once about the words 'abnormal' and 'extremal'. A corank k abnormal extremal point is in fact a corank k critical point of the mapping F. We avoid introducing an extra word 'rigidity' for phenomenon of isolatedness of point $\hat{x}$ on the level $F^{-1}(0)$ of the mapping F, the phenomenon, we suppose to deal with in this Section.

Certainly for $\hat{x}$ to be isolated it is necessary to be critical, since otherwise locally in a small neighborhood of $\hat{x}$ the level $F^{-1}(0)$ is Banach manifold (without isolated points). We are going to set necessary/sufficient conditions for isolatedness of critical point $\hat{x}$ on the critical level $F^{-1}(0)$ of the mapping F.

Assuming that F is twice Frechet differentiable at the point $\hat{x}$ we involve into consideration the Hessian of F at the point $\hat{x}$ (see [9]). It is quadratic mapping $F''(\hat{x}) : \ker F'|_{\hat{x}} \to \operatorname{coker} F'|_{\hat{x}}$. One can represent it as a bundle of quadratic forms

$$\lambda \to \lambda F''(\hat{x})(\xi, \xi), \ \xi \in \ker F'|_{\hat{x}}, \ \lambda \perp \operatorname{Im} F'|_{\hat{x}},$$

with the domain $\ker F'|_{\hat{x}}$.

We define *index* and *nullity* of $(\hat{x}, \lambda)$ as negative index and dimension of the kernel of the quadratic form $\lambda F''(\hat{x})(\xi, \xi)$ on $\ker F'|_{\hat{x}}$. *Index of critical point* $\hat{x}$ is $\min\{\operatorname{ind} \lambda F'' | \lambda \perp \operatorname{Im} F'|_{\hat{x}}\}$.

We will show now, that index and nullity provide essential information about local structure of the critical level $F^{-1}(0)$.

Theorem 9.1 (Isolated Points on Critical Levels: Necessary Condition) *Assume, that X is a Banach space and let $\hat{x} \in X$ be a corank m critical point for the mapping $F : X \to R^k$, which is twice Frechet differentiable at $\hat{x}$. Then for $\hat{x}$ to be an isolated point of the set $F^{-1}(0)$, its index can not exceed $m - 1$.* □

Proof of Theorem 9.1. We assume without loss of generality, that $\hat{x}$ is the origin of X. We denote by D and h correspondingly the differential and the Hessian of the mapping F at the origin. Suppose, that for any $\lambda \in (\operatorname{Im} D)^{\perp}$ index of the quadratic form λh on $\ker D$ is $\geq m$. We are going to prove that then $\hat{x} = 0$ is not isolated point of the set $F^{-1}(0)$.

The equation $F(x) = 0$ can be represented as a system $f(y, z) = 0, \ g(y, z) = 0$, where $(y, z) = x$ is such splitting of x, that: i) z coordinatizes $\ker D$, $\partial F / \partial z = 0$;

$$\text{ii)} \ \dim f = \dim y = \operatorname{rank} D = k - m, \ \operatorname{rank} \partial F / \partial y|_0 = \operatorname{rank} \partial f / \partial y|_0 = \operatorname{rank} D$$

Then in virtue of Implicit Function Theorem the equation $f(y, z) = 0$ can be resolved uniquely w.r.t. $y : \ y = y(z)$. Substituting $y(z)$ into the equation $g(y, z) = 0$ we obtain an equation

$\varphi(z) = g(y(z), z) = 0$. Obviuously $x = 0$ is an isolated point of the set $F^{-1}(0)$, if and only if $z = 0$ is an isolated point of $\varphi^{-1}(0)$. Let us note, that $\varphi'(0) = 0$ and hence we may investigate now the mapping φ, whose differential vanishes at the origin. To avoid additional notation we will assume, instead of it, that $D = F'|_0 = 0$, and then $h = F''|_0$ is a quadratic mapping of X into R^m. Again we assume, that for any nonzero $\lambda \in R^{m^*}$ index of the quadratic form λh is $\geq m$.

Now we will get rid of infinite-dimensional space X.

Lemma 9.2 *Under the conditions of the Theorem 9.1 there exists a finite-dimensional subspace $W \subset X$, such that for any nonzero $\lambda \in R^{m^*}$ index of the quadratic form $\lambda h|_W$ is $\geq m$.* □

Proof of Lemma 9.2. For any unit covector $\bar{\lambda} \in R^{m^*}$, there exists a $m-$dimensional subspace $W_{\bar{\lambda}} \subset X$, such that the restriction $\bar{\lambda} h|_{W_{\bar{\lambda}}}$ is negative definite. For all λ's from some small neighborhood $\Omega_{\bar{\lambda}}$ of $\bar{\lambda}$ the quadratic forms $\lambda h|_{W_{\bar{\lambda}}}$ are also negative definite. Choosing a finite covering of the sphere $\|\lambda\| = 1$ by corresponding neighborhoods $\Omega_{\bar{\lambda}^1}, \dots \Omega_{\bar{\lambda}^s}$ we may take $W = W_{\bar{\lambda}^1} + \cdots W_{\bar{\lambda}^s}$. ■

From now on we consider W in place of X or, all the same, assume $\dim X < \infty$.

The following statement enables us to investigate the quadratic mapping h instead of F.

Lemma 9.3 *If the cone $(h)^{-1}(0)$ contains a regular point of the quadratic mapping $h : X \to R^m$, then 0 is not isolated point of the set $F^{-1}(0)$.* □

Proof. If $y \in h^{-1}(0)$ is a regular point of h, then there exists a $m-$dimensional subspace $Z \subset X$, such that $h|_{y+Z} : (y + Z) \to R^m$ is local diffeomorphism at y. Since h is homogeneous, then the same holds for all points $\eta^2 y,\ \eta \neq 0$.

Consider the mapping $\phi_\epsilon(z) = h(y + \epsilon z)$, where z belongs to the unit sphere $S^{m-1} \subset Z$. Obviously $h(\eta y + \eta\epsilon z) = \eta^2 \phi_\epsilon(z)$ and, for small enough $\epsilon > 0$, the topological degree of the mapping $\phi_\epsilon / \|\phi_\epsilon\| : S^{m-1} \to S^{m-1}$ is +1 or -1. Since the differentials of h at the points ηy are nondegenerate, then $\exists a > 0$, such that for small enough $\epsilon > 0$, $\forall z \in S^{n-1}$: $\|h(\eta y + \eta\epsilon z)\| \geq a\eta^2\epsilon$.

On the other side

$$\|F(\eta y + \eta\epsilon z) - h(\eta y + \eta\epsilon z)\| = o(\eta^2)$$

and therefore for some $\epsilon > 0$ and small enough $\eta > 0$ topological degree of the mapping

$$z \longrightarrow F(\eta y + \eta\epsilon z)/\|F(\eta y + \eta\epsilon z)\|$$

is +1 or -1. Hence for every small enough $\eta > 0$ the equation $F(\eta y + \eta\epsilon z) = 0$ has a solution z_η belonging to the unit ball $B^n \subset R^n$ and therefore 0 is not isolated point of the set $F^{-1}(0)$. ■

This Lemma allows us to deal with the quadratic mapping h instead of F. The conclusion of the Theorem 4.1 is implied by the following

Proposition 9.4 *Let $P : X \to R^m$ be quadratic mapping ($\dim X < \infty$), such that* $\operatorname{ind} \lambda P \geq m$, $\forall \lambda \in R^{m^*} \setminus 0$. *Then $P^{-1}(0)$ contains regular point of the mapping P.* □

Proof. Without loss of generality we may assume, that $P'|_x = 0$ if and only if $x = 0$. Indeed otherwise the condition $P'|_x = 0$ means, that x lies in the intersection of the kernels of the (quadratic forms, which are) components of P, and we may take quotient of X w.r.t. to this intersection.

Let us start induction w.r.t. m. For $m = 1$ the theorem was proved in [7]. Taking $m > 1$ we will treat separately two cases.

i) $\underline{P^{-1}(0) \neq \{0\}}$. Let $P(y) = 0$ for some $y \neq 0$. Then $P'|_y \neq 0$ and $\lambda P''|_y$ coincides with the restriction of λP onto $\ker P'|_y$ ($\lambda \perp \operatorname{Im} P'|_y$). Conditions of the Proposition imply $\operatorname{ind} \lambda P''|_y \geq \operatorname{codim} \operatorname{Im} P'|_y,\ \forall \lambda \perp \operatorname{Im} P'|_y$. Since $\dim \operatorname{coker} P'|_y < m$, then according to the inductive assumption the inverse image $(P''|_y)^{-1}(0)$ contains regular point and hence $P^{-1}(0)$ contains regular point.

ii) $\underline{P^{-1}(0) = \{0\}}$. Then Im$P$ is a *closed* cone. Applying inductive assumption to the Hessians $P''|_y$ for any $y \neq 0$ we obtain, that $\forall z \in \mathrm{Im}P \setminus 0$ the inverse image $P^{-1}(z)$ contains regular point of P and therefore Im$P \setminus 0$ is *open*. This means that Im$P = R^m$.

Let S be unit sphere in X. Then $P/\|P\||_S : S \to S^{m-1}$ is a surjective mapping. In virtue of Sard theorem there exists regular value v of this mapping. If $P(x) = av$ for some $a > 0$, then rank$(P|_S)'_x \geq m-1$. Let $a_v = \min\{a > 0 | av \in P(S)\}$; $a_v > 0$, since $P^{-1}(0) = 0$. Let $x_v \in S$ and $P(x_v) = a_v v$. The pair (a_v, x_v) is point of local minimum and normal extremal point for the following problem of relative extremum:

$$a \to \min, \; P(x) - av = 0.$$

Standard 2nd-order necessary optimality condition for this problem provides existence of $\lambda \in R^{m^*} \setminus 0$, such that:

$$\lambda v \geq 0, \; \lambda (P|_S)'|_{x_v} = 0, \; \lambda (P|_S)''|_{x_v} \geq 0.$$

Direct computation gives

$$1/2\lambda (P|_S)''|_{x_v}(y) = \lambda P(y) - |y|^2 \lambda P(x_v),$$

and hence $\lambda P(y) \geq 0$ for all $y \in N = \{y \mid y \perp x_v, P'|_{x_v} y = 0\}$. Obviously N is a linear subspace of codimension m in X. Since $\lambda P(x_v) = \lambda v \geq 0$ and x_v is orthogonal and $P-$orthogonal to N , then λP is nonnegative on $N \oplus \mathrm{span}\{x_v\}$ and hence ind$\,\lambda P \leq m-1$, i.e. we come to a contradiction, which **finishes the proofs of the Proposition 9.4 and Theorem 9.1.** ■

Now we are going to derive from the previous theorem the necessary rigidity condition for abnormal geodesics, which was established in the Section 4.

Proof of the Theorem 4.1. We consider $\{x = (t, u(\cdot)) \in R \times L^r_\infty[0,T] | \; |u(t)| \equiv 1\}$, $\hat{x} = (T, \hat{u}(\cdot))$, and $F = F(t, u(\cdot))$ be the (time$\times$input)/state mapping. Since our consideration is local, we may coordinatize small neighborhood of $q^1 = F((T, \hat{u}(\cdot))$ in M by R^n and small neighborhood of $(T, \hat{u}(\cdot))$ by $X = R \times L^{r-1}_\infty[0,T]$. The (time$\times$input)/state mapping is not smooth w.r.t. time parameter t but becomes $C^\ell-$smooth if we restrict it on the space of $C^\ell-$smooth controls $u(\cdot)$. Obviously the hessian of this restriction coincides with the 2nd variation (4.9)-(4.10) and index of the critical point $\hat{x} = (T, \hat{u}(\cdot))$ for this restriction coincides with the one from the Definition 4.4.

Now the Theorem 4.1 follows from the Theorem 9.1. ■

Theorem 9.5 (Isolated Points on Critical Levels: Sufficient Condition) *Let the Banach space X be densely embedded into separable Hilbert space H : $X \hookrightarrow H$. Let a mapping $F : X \to R^m$ be Frechet differentiable at $\hat{x} \in X$ which is a critical point of F : $\lambda F'(\hat{x}) = 0$ for some $\lambda \in R^{m^*} \setminus \{0\}$. Assume that for F the following Taylor formula at $\hat{x}$*

$$F(\hat{x} + x) - F(\hat{x}) = F'(\hat{x})x + F''(\hat{x})(x,x) + o(\|x\|_X^2)$$

holds where $F''(\hat{x})(x,x)$ is continuous quadratic mapping: $F'' : X \to R^m$. Let us choose a complementary space Z to $\ker F'(\hat{x})$ *and represent any $x \in X$ as: $x = z + \xi$, $z \in Z$, $\xi \in$* $\ker F'(\hat{x})$. *If:*

$$\text{i) } \|F(\hat{x} + x) - F(\hat{x}) - F'(\hat{x})x\| = o(1)\|x\|_H \text{ as } \|x\|_X \to 0; \tag{9.2}$$

ii) the quadratic form $\lambda F''(\hat{x})(\xi, \xi)$ admits continuous extension from $\ker F'(\hat{x})$ *onto its completion in H and is $H-$positive definite on this completion, i.e. for some $\gamma > 0$*

$$\lambda F''(\hat{x})(\xi, \xi) \geq 2\gamma \|\xi\|_H^2 \tag{9.3}$$

for $\xi \in \ker F'(\hat{x})$;

$$\text{iii) } \|\lambda F(\hat{x} + \xi) - (\lambda F(\hat{x}) + \frac{1}{2}\lambda F''(\hat{x})(\xi, \xi))\| = o(1)\|\xi\|_H^2, \text{ as } \|\xi\|_X \to 0, \tag{9.4}$$

for $\xi \in \ker F'(\hat{x})$;

$$\text{iv) } \|\lambda F(\hat{x} + x) - (\lambda F(\hat{x}) + \frac{1}{2}\lambda F''(\hat{x})(\xi, \xi))\| = O(1)\|x\|_H\|z\| + o(1)\|\xi\|_H^2, \text{ as } \|x\|_X \to 0, \tag{9.5}$$

then $\hat{x}$ is an isolated point of the level set $F^{-1}(F(\hat{x}))$. □

Proof of the Theorem 9.5. Without loss of generality we may assume, that $F(\hat{x}) = 0$ and $\hat{x}$ is the origin of X. We are going to establish, that $\|F(x)\| \geq \mu \|x\|_H^2$ for some $\mu > 0$ and all x from some small neighborhood of the origin in X.

Let us take for Z the finite-dimensional orthogonal (in H) complement to $\ker F'(0)$; $F'(0)$ maps Z isomorphically onto the image $F'(0)X$ and for some $c > 0$

$$\|F'(0)z\| \geq c\|z\| \ \forall z \in Z. \tag{9.6}$$

Defining $N = \{y \in R^m | \lambda \cdot y = 0\}$ we choose a vector $\eta \in R^m$ such, that $\lambda \cdot \eta = 1$. Then $R^m = R\eta + N$ and $\mathrm{Im} F'(0) \subseteq N$.

If $x = z + \xi$ then using Hadamard lemma we may present $F(x)$ as

$$F(x) = \Phi(\xi) + F'(0)z + A(x)z.$$

In virtue of (9.2), $\|\Phi(\xi) + A(x)z\| = o(1)(\|\xi\|_H + \|z\|)$ as$\|x\|_X \to 0$.

Let us consider the projections of $F(x)$ onto the vector η and the subspace N; they are $\lambda \cdot (\Phi(\xi) + A(x)z)\eta$ and $R(x) = F'(0)z + \Phi_N(\xi) + A_N(x)z$ correspondingly.

Fixing arbitrarily small $\epsilon > 0$ we may choose a small neighborhood V in X such that for some positive k and $x \in V$:

$$\|\Phi_N(\xi) + A_N(x)z\| \leq \epsilon \|x\|_H, \ |\lambda \cdot (\Phi(\xi) - \frac{1}{2}\lambda F''(0)(\xi,\xi) + A(x)z)| \leq k\|x\|_H \|z\| + \epsilon \|\xi\|_H^2.$$

It follows from (9.6) that

$$\|R(x)\| \geq \max(0, (c-\epsilon)\|z\| - \epsilon\|\xi\|_H), \ \forall x \in V,$$

$$|\lambda(\Phi(\xi) + A(x)z)| \geq \max(0, (\gamma - \epsilon)\|\xi\|_H^2 - k\|x\|_H\|z\|).$$

Putting $c = c - \epsilon, \ \gamma = \gamma - \epsilon$ we obtain

$$\|F(x)\| \geq \frac{1}{\sqrt{2}}(\max(0, c\|z\| - \epsilon\|\xi\|_H)) + \max(0, \gamma\|\xi\|_H^2 - k\|\xi\|_H\|z\|).$$

Without loss of generality we may assume that $k(1 + 4\epsilon/c)4\epsilon/c \leq \gamma/2$.

Now if $c\|z\| \geq 4\epsilon\|\xi\|_H$ then

$$\|F(x)\| \geq \frac{1}{\sqrt{2}}(\frac{c}{2}\|z\| + \epsilon\|\xi\|_H) \geq \alpha(\epsilon, c)\|x\|_H^2.$$

with $\alpha(\epsilon, c) > 0$.

Otherwise if $c\|z\| \leq 4\epsilon\|\xi\|_H$ then

$$\|F(x)\| \geq \frac{1}{\sqrt{2}}(\gamma\|\xi\|_H^2 - k\|x\|_H\|z\|) \geq \frac{1}{\sqrt{2}}(\gamma\|\xi\|_H^2 - \|\xi\|_H^2 k(1 + 4\epsilon/c)4\epsilon/c) \geq$$
$$\geq (\gamma/2\sqrt{2})\|\xi\|_H^2 \geq \beta(\gamma, c, \epsilon)\|x\|_H^2$$

with $\beta(\gamma, c, \epsilon) > 0$. ■

Basing on the Theorem 9.5 we shall prove the Theorem 4.8, which provides sufficient rigidity condition for abnormal geodesics.

Proof of the Theorem 4.8. We will verify the assumptions of the Theorem 9.5 for the time×input/state mapping F. Since our consideration is local i.e. regards small neighborhoods of $q^1 \in M$ of $(T, \hat{u}(\cdot)) \in \{(t, u(\cdot)) \in R \times L_\infty^r[0,T] | |u(t)| \equiv 1\}$ then coordinatizing these neighborhoods by R^n and $X = R \times L_\infty^{r-1}[0,T]$ correspondingly taking $H = R \times H_{-1}^{r-1}[0,T]$, $x = (t, u(\cdot))$; $\hat{x} = (T, \hat{u}(\cdot)), \lambda = \hat{\psi}_T$ and $F = F(t, u(\cdot))$ being the (time×input)/state mapping. If $(T, \hat{u}(\cdot))$ enters an abnormal geodesic $(\hat{u}(\cdot), \hat{q}(\cdot), \hat{\psi}(\cdot), T)$, then $(T, \hat{u}(\cdot))$ is a critical point of F and $\hat{\psi}_T \in R^{n^*}$ (see above) is an annihilator of $\mathrm{Im} F'|_{(T, \hat{u}(\cdot))}$.

Let us put

$$\|(t, u(\cdot))\|_\infty = |t| + \|u(\cdot))\|_{L_\infty}, \ \|(t, u(\cdot))\|_{-1} = (|t|^2 + \|u(\cdot))\|^2_{H_{-1}})^{1/2}$$

and extend $\hat{u}(t)$ smoothly onto $[0, T+1]$. The condition (9.3) is fullfilled by virtue of the positive definiteness of the 2nd variation. To verify other conditions it is useful to introduce another representation of F. Recall that

$$F(T+\theta, u(\cdot)) = q^0 \circ \overrightarrow{\exp} \int_0^{T+\theta} G\hat{u}(\tau)d\tau \circ \overrightarrow{\exp} \int_0^{T+\theta} Y_{T+\theta,\tau}u(\tau)d\tau =$$
$$= q^1 \circ \overrightarrow{\exp} \int_T^{T+\theta} G\hat{u}(\tau)d\tau \circ \overrightarrow{\exp} \int_0^{T+\theta} Y_{T+\theta,\tau}u(\tau)d\tau,$$

where $Y_{t,\tau}v = \overrightarrow{\exp} \int_t^\tau \operatorname{ad} G\hat{u}(\xi)d\xi Gv$.

Now we shall transform the chronological exponential $\overrightarrow{\exp} \int_0^{T+\theta} Y_{T+\theta,\tau}u(\tau)d\tau$ by means of the *integration by parts formula for a chronological exponential*, established in [20]. We derive:

$$F(T+\theta, u(\cdot)) = q^1 \circ \overrightarrow{\exp} \int_T^{T+\theta} G\hat{u}(\tau)d\tau \circ \tag{9.7}$$
$$\overrightarrow{\exp} \int_0^{T+\theta} ((I - \int_0^1 e^{(1-\eta)\operatorname{ad} Y_{T+\theta,\tau}v(\tau)}d\eta)Y_{T+\theta,\tau}u(\tau) - \int_0^1 e^{(1-\eta)\operatorname{ad} Y_{T+\theta,\tau}v(\tau)}d\eta \dot{Y}_{T+\theta,\tau}v(\tau))d\tau \circ e^{Gv(T+\theta)}$$

(here $\dot{Y}$ denotes the partial derivative $\partial Y_{t,\tau}/\partial\tau$).

Taking Volterra expansions for $\overrightarrow{\exp} \int_T^{T+\theta} G\hat{u}(\tau)d\tau$ and for the ordinary and chronological exponentials in the last formula we derive

$$F(T+\theta, \hat{u}(\cdot) + u(\cdot)) = q^1 + q^1 \circ (\hat{f}_T\theta - \int_0^T \dot{Y}_{T+\theta,\tau}v(\tau)d\tau + Gv(T))$$
$$+ o(1)(|\theta| + |v(T)| + \|v(\cdot)\|_{L_2}) \text{ as } \|(\theta, u(\cdot))\|_\infty \to 0$$

from where the estimate (9.2) follows directly.

To verify other estimates we will multiply (9.7) by $\lambda = \hat{\psi}_T$ and simplify it getting rid of the terms which are more than quadratic in $v(\cdot)$ and therefore admit an estimate $o(1)\|u(\cdot)\|^2_{-1}$. We obtain

$$\hat{\psi}_T(F(T+\theta, \hat{u}(\cdot) + u(\cdot)) - q^1) = \hat{\psi}_T q^1 \circ \overrightarrow{\exp} \int_T^{T+\theta} G\hat{u}(\tau)d\tau \circ$$
$$\circ(-I + \overrightarrow{\exp} \int_0^{T+\theta} -[Y_{T+\theta,\tau}v(\tau), Y_{T+\theta,\tau}u(\tau)] - \dot{Y}_{T+\theta,\tau}v(\tau) + [\dot{Y}_{T+\theta,\tau}v(\tau), Y_{T+\theta,\tau}v(\tau)]d\tau \circ e^{Gv(T+\theta)})$$
$$+ o(1)\|(\theta, u(\cdot)\|^2_{-1}$$

Taking into account the continuity of $\hat{f}_T$ w.r.t. τ, the equalities:

$$\forall u, v \in R^r \ \hat{\psi}_T[Y_{T,\tau}v, Y_{T,\tau}u](q^1) = 0; \ \hat{\psi}_T G(q^1)u = 0; \ \hat{\psi}_T Y_{T,\tau}(q^1)u = \hat{\psi}_T \dot{Y}_{T,\tau}(q^1)v = 0$$

and collecting in the rest term the terms of order > 2 in $(\theta, u(\cdot), v(\cdot))$ in the Volterra expansions of the chronological exponentials we derive:

$$\hat{\psi}_T(F(T+\theta, \hat{u}(\cdot) + u(\cdot)) - q^1) =$$
$$= \hat{\psi}_T q^1 \circ (\hat{f}_T \circ \hat{f}_T\theta^2/2 - \theta \int_0^T \partial/\partial\theta|_0 \dot{Y}_{T+\theta,\tau}v(\tau)d\tau + \int_0^T [\dot{Y}_{T,\tau}v(\tau), Y_{T,\tau}v(\tau)]d\tau + \hat{f}_T\theta \int_0^T \dot{Y}_{T,\tau}v(\tau)d\tau$$
$$+ \int_0^T \int_0^t \dot{Y}_{T,\tau}v(\tau)d\tau \circ \dot{Y}_{T,t}v(t)dt + \theta\hat{f}_T Gv(T) + \frac{1}{2}Gv(T) \circ Gv(T) + O(\|(\theta, u(\cdot))\|_\infty \|(\theta, u(\cdot))\|^2_{-1}).$$

Therefore we have established for the time×input/state mapping F the estimates (in the notations of the previous theorem):

$$|\lambda F(\hat{x}+x)-\lambda F(\hat{x})|=O(1)\|x\|_H^2,\ \text{as}\ \|x\|_X\to 0 \tag{9.8}$$

$$\|\lambda F(\hat{x}+x)-(\lambda F(\hat{x})+\frac{1}{2}\lambda F''(\hat{x})(x,x))\|=O(\|x\|_X\|x\|_H^2),\ \text{as}\ \|x\|_X\to 0 \tag{9.9}$$

and the continuity of the quadratic form $\lambda F''(\hat{x})(x,x)$ on H. This implies (9.4).

To derive (9.5) from (9.8)-(9.9) it is enough to establish the estimate $|\lambda F''(\hat{x})(x,x)-\lambda F''(\hat{x})(\xi,\xi)|=O(1)\|x\|_H\|z\|$ which follows from the continuity of $\lambda F''(\hat{x})(x,x)$ in H. ■

References

[1] A.A. Agrachev, *Quadratic Mappings in Geometric Control Theory,* in: Itogi Nauki i Tekhniki; Problemy Geometrii, VINITI, Acad. Nauk SSSR, Moscow, Vol.20,pp.11-205,1988. English transl. in J.Soviet Math.,Vol.51,pp.2667-2734,1990.

[2] A.A. Agrachev, *Topology of Quadratic Mappings and Hessians of Smooth Mappings,* in: Itogi Nauki i Tekhniki; Algebra, Topologia, Geometria; VINITI, Acad. Nauk SSSR, Vol.26,pp.85-124,1988.

[3] A.A. Agrachev, *The Second-Order Optimality Condition in the General Nonlinear Case,*Matem Sbornik, Vol. 102, pp.551-568, 1977. English transl. in: Math. USSR Sbornik,Vol. 31, 1977.

[4] A.A.Agrachev and R.V.Gamkrelidze, *Second-Order Optimality Condition for the Time-Optimal Problem,* Matem Sbornik, Vol.100,pp.610-643,1976. English transl. in: Math. USSR Sbornik,Vol.29,pp.547-576,1976.

[5] A.A.Agrachev and R.V.Gamkrelidze, *Exponential Representation of Flows and Chronological Calculus,* Matem. Sbornik, Vol.107, pp.467-532,1978. English transl. in: Math. USSR Sbornik, Vol.35, pp.727-785,1979.

[6] A.A. Agrachev, R.V. Gamkrelidze and A.V. Sarychev, *Local Invariants of Smooth Control Systems,* Acta Applicandae Mathematicae, Vol.14,pp.191-237,1989.

[7] A.A.Agrachev and A.V. Sarychev, *On Abnormal Extremals for Lagrange Variational Problems,* to appear in J.Mathematical Systems, Estimation and Control.

[8] V.I. Arnol'd, *Mathematical Methods of Classical Mechanics,* Springer-Verlag, New York-Berlin, 1978.

[9] V.I. Arnol'd, A.N.Varchenko and S.M.Gusein-Zade, *Singularities of Differentiable Mappings,* Vol.1,Birkhäuser,Boston,1985.

[10] B.Bonnard and I.Kupka, *Theorie de Singularites de l'Application Entree/Sortie et Optimalite des Trajectoires Singulieres dans le Probleme du Temps Minimal,* Preprint,1990.

[11] V. Gershkovich, *Engel Structures on Four Dimensional Manifolds,* University of Melbourne, Department of Mathematics, Preprint Series No. 10, 1992.

[12] B.S.Goh, *Necessary Conditions for Singular Extremals Involving Multiple Control Variables,* SIAM J.Control, Vol.4, pp.716-731,1966.

[13] V.Guillemin and S.Sternberg, *Geometric Asymptotics,* Amer.Math.Soc., Providence, Rhode Island,1977.

[14] G.W.Haynes and H.Hermes, *Nonlinear Controllability via Lie Theory,* SIAM J. on Control,Vol.8,pp.450-460,1970.

[15] H.J.Kelley, R.Kopp and H.G. Moyer, *Singular Extremals,* in: G.Leitman Ed., Topics in Optimization, Academic Press, New York,N.Y.,pp.63-101,1967.

[16] A.J.Krener, *The High-Order Maximum Principle and its Applications to Singular Extremals,* SIAM J. on Control and Optimiz.,Vol.15,pp.256-293,1977.

[17] G.Lion and M.Vergne, *The Weyl Representation, Maslov Index and Theta-series*, Birkhäuser,Boston,1980.

[18] R.Montgomery, *Geodesics, Which Do Not Satisfy Geodesic Equations*, Preprint, 1991.

[19] M.Morse, *The Calculus of Variations in the Large,* Amer.Math.Soc.,New York,NY,1934.

[20] A.V. Sarychev, *Integral representation for the trajectories of control system with generalized right-hand side* (in Russian), Differentsialnye Uravnenia,Vol.24,pp.1551-1564,1988. English transl.: Differential Equations, Vol.24, pp.1021-1031,1988.

[21] A.V.Sarychev, *The index of the second variation of the control system*, Matemat. Sbornik, Vol.113, pp.464-486, 1980. English transl. in: Math.USSR Sbornik, Vol.41, pp.383-401, 1982.

[22] H.J.Sussmann, *A Cornucopia of Abnormal SubRiemannian Minimizers. Part I: the Four-Dimensional Case*, IMA Preprint Series #1073,1992.

[23] L.C.Young, *Lectures on the Calculus of Variations and Optimal Control Theory,* Chelsea, New York, 1980.