Analysis of cusps and characterization of the existence set in an association field model on the retinal sphere

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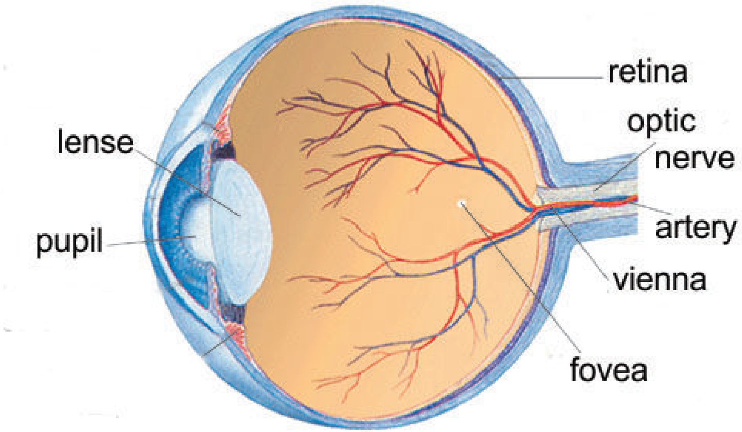
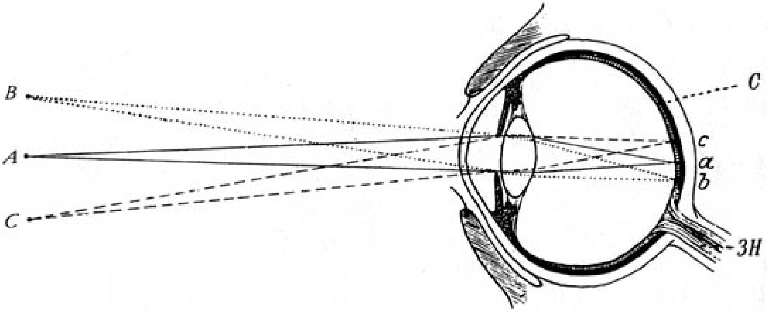
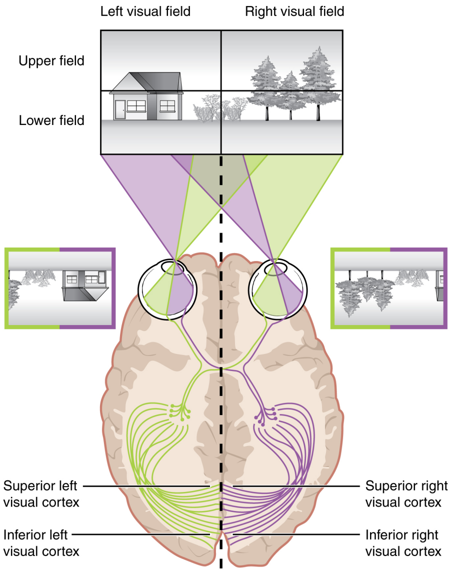
Structure of Human Eye



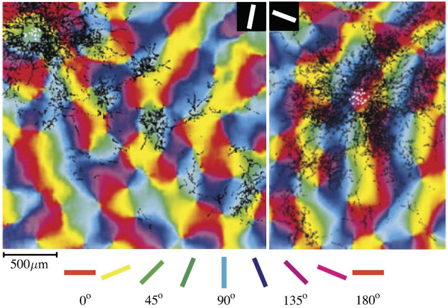
Perception of Visual Information in Human Eye



Perception of Visual Information in Human Brain

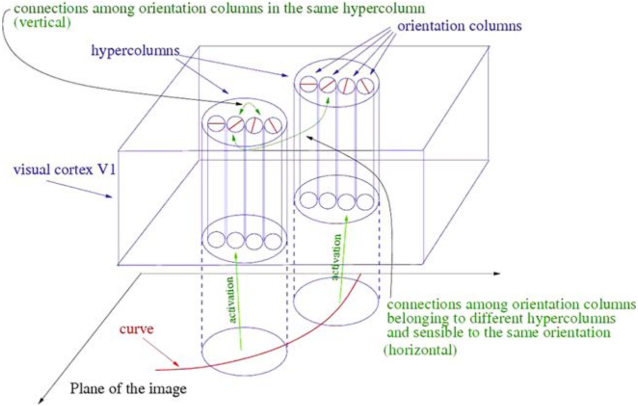


Receptive Fields in the Visual Cortex



Receptive fields in the visual cortex of many mammalians are tuned to various locations and orientations. Assemblies of oriented receptive fields are grouped together on the surface of the primary visual cortex in a pinwheel like structure. Replicated from **Bosking, W.H., et al.**, Orientation selectivity and the arrangement of horizontal connections in tree shrew striate cortex. *J. Neurosci. 17(6), 2112–2127*, 1997.

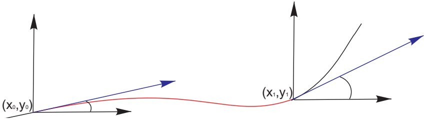
A Model of the Primary Visual Cortex V1



Replicated from **R. Duits, U. Boscain, F. Rossi, Y. Sachkov**, Association Fields via Cuspless Sub-Riemannian Geodesics in SE(2), *J Math Imaging Vis*, 2013.

Contours Completion

- Neurogeometry of vision:
    - Petitot J., The neurogeometry of pinwheels as a sub-Riemannian contact structure, 2003
    - Petitot J., Neurogeometrie de la vision. Modeles mathematiques et physiques des architectures fonctionelles, 2008
- Variational principle: a completed arc must have minimum length in the space (x, y, θ):



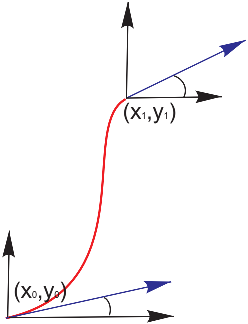
$$\int \sqrt{\dot{x}^2 + \dot{y}^2 + \xi^2\,\dot{\theta}^2}dt \to \min .$$

- Sub-Riemannian problem on SE(2).

Sub-Riemannian geometry problem on SE(2)

- Optimal Control Problem

$$
\begin{aligned}
&\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} = u_2,\\
&q = (x, y, \theta) \in M = \mathbb{R}^2_{x,y} \times S^1_\theta,\\
&u = (u_1, u_2) \in \mathbb{R}^2,\\
&q(0) = q_0 = (0, 0, 0),\\
&q(t_1) = q_1 = (x_1, y_1, \theta_1),\\
&l = \int_0^{t_1} \sqrt{u_1^2 + \xi^2\, u_2^2}\, dt \to \min.
\end{aligned}
$$



- Solved by Yuri Sachkov (Cut locus and optimal synthesis in the sub-Riemannian problem on the group of motions of a plane, 2010).

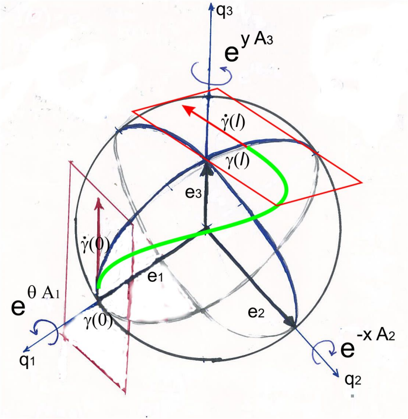
Improvement of the Flat Model

Motivations: RETINA IS NOT FLAT: BOTH IMPORTANT FOR CORTICAL MODELING AS FOR PROCESSING RETINAL IMAGES IN LONG RUN.

We consider the problem of minimizing $\int_0^l \sqrt{1+\xi^2 k_g^2(s)}\, ds$ for a curve on a sphere with fixed boundary points and directions. The total length l is free, s denotes the spherical arclength, and k_g denotes the geodesic curvature of the sphere. This problem is a natural extension of a model due to Petitot, Citti & Sarti, which additionally takes into account the spherical nature of the retina.

Problem statement ($P_{\mathbf{curve}}$)

$$n = (n_1, n_2, n_3)^T \in S^2, \gamma(s) = n(s),$$
$$\gamma(0) = (1,0,0)^T, \gamma(l) = \gamma_1,$$
$$\gamma'(0) = (0,0,1)^T, \gamma'(l) = \gamma_1',$$
$$\mathrm{E}(\gamma(\cdot)) =$$
$$\int_0^l \sqrt{1 + \xi^2 k_g^2(s)}\, ds \to \min,$$
$$s = \int_0^s 1 d\sigma =$$
$$\int_0^s \sqrt{\sum_{i=1}^{3} n_i'(\sigma)^2 d\sigma},$$
$$k_g(s) = \gamma''(s)(\gamma(s) \times \gamma'(s)).$$



Results and Plans

Results:

- Lift $\mathbf{P}_{\mathbf{curve}}$ to Sub-Riemannian Problem on SO(3),
- Control System in Quaternion, Matrix and Vector Representations,
- Hamiltonian System for Geodesics,
- Symmetries of Exponential Mapping,
- Integration of Hamiltonian System in Spherical Arclength,
- Evaluation of Cusp Time s_{max} in Elliptic Case $\xi > 1$.

Plans:

- Projections of Stationary Curves from $\mathbf{P}_{\mathbf{mec}}$ to $\mathbf{P}_{\mathbf{curve}}$,
- Plot of Existence Cones of Cuspless Geodesics.

$\mathbf{P}_{\mathbf{mec}}$: Lift $\mathbf{P}_{\mathbf{curve}}$ to SR-Problem on SO(3)

- $\mathrm{SO}(3) \ni R(x, y, \theta) = \mathrm{e}^{yA_3}\,\mathrm{e}^{-xA_2}\,\mathrm{e}^{\theta A_1} =$

$$= \begin{pmatrix} \mathrm{cx\,cy} & -\mathrm{cy\,sz\,sx} - \mathrm{cz\,sy} & \mathrm{sz\,sy} - \mathrm{cz\,cy\,sx} \\ \mathrm{cx\,sy} & \mathrm{cz\,cy} - \mathrm{sz\,sx\,sy} & -\mathrm{cy\,sz} - \mathrm{cz\,sx\,sy} \\ \mathrm{sx} & \mathrm{cx\,sz} & \mathrm{cz\,cx} \end{pmatrix},$$

$$\begin{cases} \mathrm{cx} = \cos x, \\ \mathrm{sx} = \sin x, \end{cases} \qquad \begin{cases} \mathrm{cy} = \cos y, \\ \mathrm{sy} = \sin y, \end{cases} \qquad \begin{cases} \mathrm{cz} = \cos \theta, \\ \mathrm{sz} = \sin \theta, \end{cases}$$

  where $(x, y) \in \mathrm{S}^2$, $\theta \in \mathrm{S}^1$.

- Projection to $\mathbf{P}_{\mathbf{curve}}$:

$$\gamma(s) = R(s)e_1, \qquad \gamma'(s) = R(s)e_3.$$

P_{mec}: Sub-Riemannian Problem on SO(3)

The Lie algebra of left-invariant vector fields on the Lie group SO(3)

$$\mathrm{L} = \mathrm{span}(X_1, X_2, X_3), \qquad \begin{cases} X_1 = -R\,A_2, \\ X_2 = R\,A_1, \\ X_3 = [X_2, X_1] = -R\,A_3, \end{cases}$$

where $R \in \mathrm{SO}(3)$ and A_i, $i = 1, 2, 3$ are generators of Lie algebra $\mathrm{so}(3) = \mathrm{span}(A_1, A_2, A_3)$:

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, A_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\mathrm{P}_{\mathrm{mec}}$: Sub-Riemannian Problem on SO(3)

- Control System
$$\dot{R} = u_1 X_1 + u_2 X_2.$$
- Boundary Conditions
$$R(0) = \mathrm{Id} \Leftrightarrow \begin{cases} x(0) = 0, \\ y(0) = 0, \\ \theta(0) = 0, \end{cases} \qquad R(t_1) = R_1 \Leftrightarrow \begin{cases} x(t_1) = x_1, \\ y(t_1) = y_1, \\ \theta(t_1) = \theta_1. \end{cases}$$
- Cost Functional
$$l = \int_0^{t_1} \sqrt{u_1^2 + \xi^2 u_2^2}\, dt \to \min .$$
-
$$R \in \mathrm{SO}(3),\ (u_1, u_2) \in \mathbb{R}^2,\ \xi > 0.$$

P_{mec} in Vector Representation

- By collecting coefficients at $\frac{\partial R}{\partial x}$, $\frac{\partial R}{\partial y}$, $\frac{\partial R}{\partial \theta}$ we can represent the control system in the following form:

$$\dot{\nu} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos\theta \\ -\sec x \sin\theta \\ \sin\theta \tan x \end{pmatrix} u_1 + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u_2,$$

where $\nu = (x, y, \theta)^T$.

$\mathrm{P_{mec}}$ in Quaternion Representation

- Components of Quaternion

$$\begin{cases} q_0 = \frac{1}{\delta}((\mathrm{cz}+1)(\mathrm{cx}+1)(\mathrm{cy}+1) - \mathrm{sz}\,\mathrm{sx}\,\mathrm{sy}), \\ q_1 = \frac{1}{\delta}(\mathrm{sz}(\mathrm{cx}+1)(\mathrm{cy}+1) + (\mathrm{cz}+1)\,\mathrm{sx}\,\mathrm{sy}), \\ q_2 = \frac{1}{\delta}(\mathrm{sz}(\mathrm{cx}+1)\,\mathrm{sy} - (\mathrm{cz}+1)\,\mathrm{sx}(\mathrm{cy}+1)), \\ q_3 = \frac{1}{\delta}(\mathrm{sz}\,\mathrm{sx}(\mathrm{cy}+1) + (\mathrm{cz}+1)(\mathrm{cx}+1)\,\mathrm{sy}), \end{cases}$$

$$\text{where } \delta = 2\sqrt{2}\sqrt{\mathrm{cz}+1}\sqrt{\mathrm{cx}+1}\sqrt{\mathrm{cy}+1}.$$

- Control System

$$\dot{q} = \frac{(q_2, q_3, -q_0, -q_1)^T}{2} u_1 + \frac{(-q_1, q_0, q_3, -q_2)^T}{2} u_2.$$

- Boundary Conditions $q(0) = 1, \quad q(t_1) = q^1.$
- Cost Functional $l = \int_0^{t_1} \sqrt{u_1^2 + \xi^2 u_2^2}\, dt \to \min.$

Pontryagin Maximum Principle

- Left Invariant Hamiltonians $h_i = \langle \lambda, X_i \rangle$, $i = 1, 2, 3$
- Maximality Condition

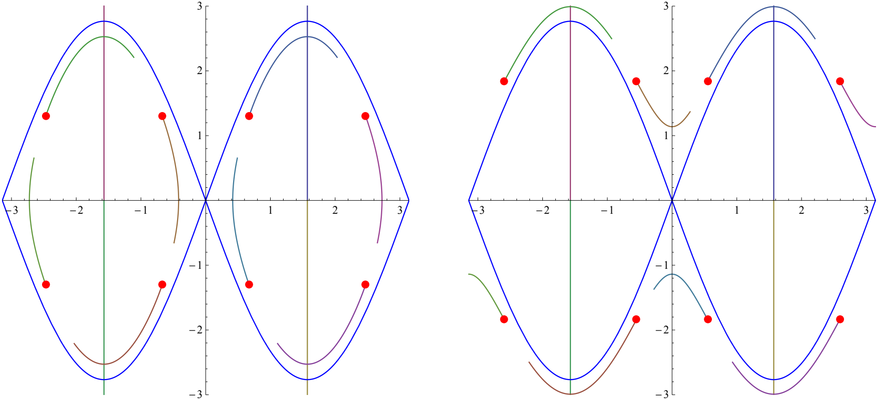
$$u_1 = h_1, \qquad u_2 = \frac{h_2}{\xi^2}.$$

- Hamiltonian System

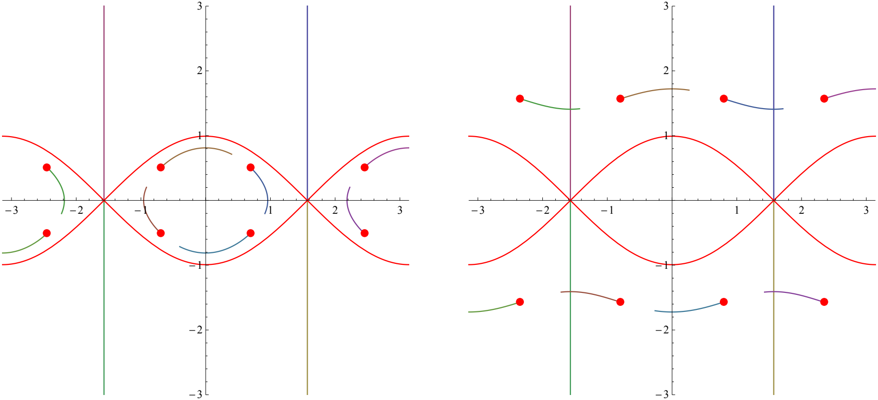
$$\begin{cases} \dot h_1 = \frac{1}{\xi^2} h_2 h_3, \\ \dot h_2 = -h_1 h_3, \\ \dot h_3 = (1 - \frac{1}{\xi^2}) h_1 h_2 \end{cases} \quad \text{— vertical part.}$$

$$\begin{cases} \dot x = \cos\theta \, h_1, \\ \dot y = -\sec x \sin\theta \, h_1, \\ \dot\theta = \sin\theta \tan x \, h_1 + \frac{h_2}{\xi^2} \end{cases} \quad \text{— horizontal part.}$$

Symmetries of trajectories of pendulum for $\xi < 1$



Symmetries of trajectories of pendulum for $\xi > 1$



Solution in Spherical Arclength

For any $s \in [0, \mathsf{s}_{\max})$, $h_1(0) > 0$ the vertical subsystem is equivalent to the following linear system:

$$\begin{cases} h_1 = \frac{ds}{dt} > 0, & \\ h_2'(s) = -h_3(s), & h_2(0) = h_2^0, \\ h_3'(s) = (1 - \frac{1}{\xi^2})h_2(s), & h_3(0) = h_3^0, \end{cases}$$

which has the following solution for all $\xi > 0$:

$$\begin{cases} h_2(s) = h_2(0)\cosh\left(\frac{\sqrt{1-\xi^2}}{\xi}s\right) - h_3(0)\frac{\xi}{\sqrt{1-\xi^2}}\sinh\left(\frac{\sqrt{1-\xi^2}}{\xi}s\right), \\ h_3(s) = h_3(0)\cosh\left(\frac{\sqrt{1-\xi^2}}{\xi}s\right) - h_2(0)\frac{\sqrt{1-\xi^2}}{\xi}\sinh\left(\frac{\sqrt{1-\xi^2}}{\xi}s\right), \\ h_1(s) = \sqrt{1 - \frac{h_2^2(s)}{\xi^2}}, \end{cases}$$

Connection Between Elliptic and Hyperbolic Cases

$$\chi^2 = \frac{1}{\xi^2} - 1 \quad \Leftrightarrow \quad \xi = \frac{1}{\sqrt{\chi^2+1}}, \qquad \chi = \sqrt{\frac{1}{\xi^2} - 1}.$$

h) $\chi \in \mathbb{R}^+$ - hyperbolic case $\xi \in (0,1)$,
l) $\chi = 0$ - linear case $\xi = 1$.
e) $\chi = \mathrm{i}a$, $a \in (0,1)$ - elliptic case $\xi > 1$.

$$\begin{cases} \tilde\chi = \frac{\mathrm{i}\chi}{\sqrt{\chi^2+1}}, & \chi \to -\mathrm{i}\frac{\tilde\chi}{\sqrt{\tilde\chi^2+1}}, \\ \tilde s = -\mathrm{i}s\sqrt{\chi^2+1}, & s \to \mathrm{i}\tilde s\sqrt{\tilde\chi^2+1}, \\ \tilde h_2(\tilde s) = \mathrm{i}h_2(s), & h_2(s) \to -\mathrm{i}\tilde h_2(\tilde s), \\ \tilde h_3(\tilde s) = -\frac{h_3(s)}{\sqrt{\chi^2+1}}, & h_3(s) \to -\frac{\tilde h_3(\tilde s)}{\sqrt{\tilde\chi^2+1}}. \end{cases}$$

Evaluation of s_{max} in Elliptic Case

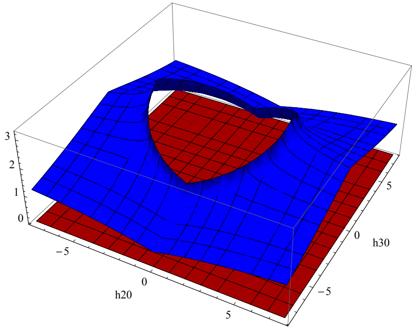
To compute $\mathrm{s}_{\mathrm{max}}$ we need to find the minimal positive root of equation $h_1(s) = 0$:

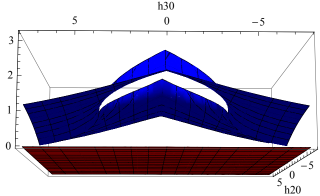
$$\mathrm{s}_{\mathrm{max}} = \min\{s > 0 | h_1(s) = 0\} = \min\{s > 0 | h_2(s) = \pm\xi\}.$$

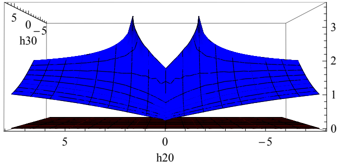
$$\mathrm{s}_{\mathrm{max}}(\xi, h_2^0, h_3^0) = \frac{\xi}{\sqrt{\xi^2 - 1}} \operatorname{arccos}\left(\frac{\xi\,(A + B)}{C}\right).$$

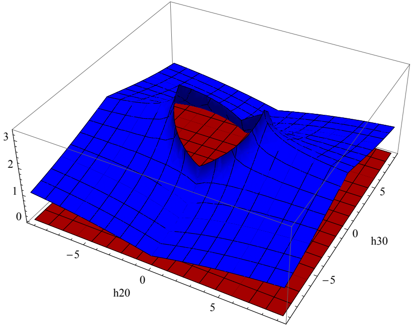
$$\begin{aligned} A &= |h_2^0|\left(1 - \xi^2\right), \\ B &= |h_3^0|\sqrt{-\left(h_2^0\right)^2 + \left(1 + \left(h_2^0\right)^2 + \left(h_3^0\right)^2\right)\xi^2 - \xi^4}, \\ C &= \left(h_3^0\right)^2\xi^2 + \left(h_2^0\right)^2\left(\xi^2 - 1\right). \end{aligned}$$

Plot of s_{max} in Elliptic Case

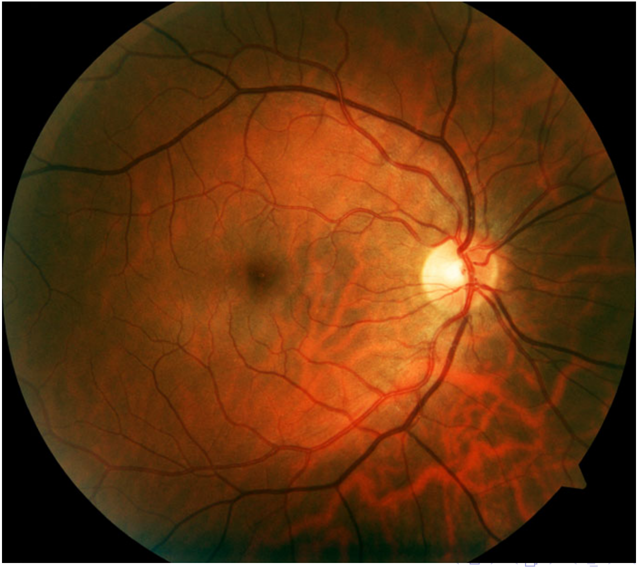








Retina Check Project



Thank you for your attention!