An Intrinsic Approach to the Control of Rolling Bodies

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Abstract

We apply principal tools of geometric control theory to an intrinsic geometric model of a pair of rolling rigid bodies. The controllability problem is solved completely: in particular, the system is globally controllable iff the bodies are not isometric. We also construct a canonical nilpotent approximation of the system, describe its symmetries, and express extremals of the corresponding optimal control problem via elliptic functions.

1 Geometric model

We consider two solid bodies in the 3-dimensional space that roll without slipping and twisting. The main questions are as follows: what motions of the bodies are realized and which of them are optimal in a certain sense.

Rather than embedding the problem into $\mathbf{R}^3$, we propose the following geometric model. Let M and $\widehat{M}$ be C^∞ two-dimensional connected oriented Riemannian manifolds (surfaces of the bodies). At each contact point of the bodies, their tangent planes are identified; this gives rise to an isometry of the tangent spaces of M and $\widehat{M}$. In this paper we deal only with orientation-preserving isometries and omit words "orientation-preseving" in order to simplify terminology. The state space of the problem is the 5-dimensional manifold

$$Q = \{ A \,:\, T_xM \to T_{\hat{x}}\widehat{M} \quad | \quad x \in M,\, \hat{x} \in \widehat{M}, \\ A \text{ an isometry} \},$$

a principal bundle over the product $M \times \widehat{M}$ with the structure group SO $(2; \mathbf{R})$.

Consider the projections from Q to M and $\widehat{M}$:

$$\pi(A) = x, \quad \hat{\pi}(A) = \hat{x}, \quad A : T_x M \to T_{\hat{x}} \widehat{M},$$
$$A \in Q, \quad x \in M, \quad \hat{x} \in \widehat{M}.$$

Denote the Gaussian curvatures of the Riemannian manifolds M and $\widehat{M}$ by k and $\hat{k}$ respectively. In the sequel we lift these curvatures from M and $\widehat{M}$ to Q:

$$k(q) = k(\pi(q)), \quad \hat{k}(q) = \hat{k}(\hat{\pi}(q)), \quad q \in Q.$$

Let $A(t) \in Q$ be an admissible curve corresponding to a motion of the rolling bodies. Then $x(t) = \pi(A(t))$ and $\hat{x}(t) = \hat{\pi}(A(t))$ are trajectories of the contact points in M and $\widehat{M}$ respectively. The condition of absence of slipping means that

$$A(t)\dot{x}(t) = \dot{\hat{x}}(t), \tag{1}$$

and the condition of absence of twisting is formalized as follows:

$$A(t)\,(\text{vector field parallel along } x(t)) = \\ (\text{vector field parallel along } \hat{x}(t))\,. \tag{2}$$

Our model ignores the state constraints that correspond to the admissibility of contact of the bodies embedded in $\mathbf{R}^3$. Notice although that if the surfaces M and $\widehat{M}$ have respectively positive and nonnegative Gaussian curvatures at a point, then their contact is locally admissible.

The admissibility conditions (1) and (2) imply that a curve $x(t) \in M$ determines completely the whole motion $A(t) \in Q$. That is, velocities of admissible motions determine a rank 2 distribution Δ on the 5-dimensional space Q. In order to write down this distribution locally, choose arbitrary smooth orthonormal frames on M and $\widehat{M}$:

$$e_1(x),\, e_2(x) \in T_x M, \quad \langle e_i, e_j \rangle_M = \delta_{ij},\ i,j = 1,2,$$
$$\hat{e}_1(\hat{x}),\, \hat{e}_2(\hat{x}) \in T_{\hat{x}} \widehat{M}, \quad \langle \hat{e}_i, \hat{e}_j \rangle_{\widehat{M}} = \delta_{ij},\ i,j = 1,2.$$

Denote by θ the angle of rotation from the frame Ae_1, Ae_2 to the frame $\hat{e}_1$, $\hat{e}_2$. Then admissible motions are integral curves of the distribution

$$\Delta_q = \operatorname{span}\,(X_1(q),\, X_2(q)), \quad q = (x, \hat{x}, \theta) \in Q$$

with

$$X_1 = e_1 + \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2 + \\ (-\alpha_1 + \hat{\alpha}_1\cos\theta + \hat{\alpha}_2\sin\theta)\frac{\partial}{\partial\theta}, \tag{3}$$

$$X_2 = e_2 - \sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2 + \\ (-\alpha_2 - \hat{\alpha}_1\sin\theta + \hat{\alpha}_2\cos\theta)\frac{\partial}{\partial\theta}, \tag{4}$$

where α_i, $\hat{\alpha}_i$ are structure constants of the frames:

$$[e_1, e_2](x) = \alpha_1(x)e_1(x) + \alpha_2(x)e_2(x),$$
$$[\hat{e}_1, \hat{e}_2](\hat{x}) = \hat{\alpha}_1(\hat{x})\hat{e}_1(\hat{x}) + \hat{\alpha}_2(\hat{x})\hat{e}_2(\hat{x}).$$

Admissible motions of the rolling bodies are trajectories of the control system

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \quad q \in Q, \quad u_1,\, u_2 \in \mathbf{R}, \tag{5}$$

i.e., admissible curves of the distribution Δ.

2 Controllability

Controllability properties of the system of two rolling bodies were studied by several authors:

- Z. Li and J. Canny [7] proved controllability for a sphere rolling on a plane, and for two spheres with different radius;
- A. Bicchi, D. Prattichizzo, and S. Sastry [4] showed controllability for bodies with regular, strictly convex and axialsymmetic surfaces rolling on a plane;
- A. Marigo and A. Bicchi [8] proved for two relatively strictly convex rolling bodies that the orbit is 2-dimensional iff the bodies are specular images of each other.

Via the geometric model, we obtain a general result.

Theorem 1 (1) *The reachable set O of system* (5) *from a point $q \in Q$ is an immersed smooth connected submanifold of Q with dimension equal to* 2 *or* 5. *Specifically*:

$$(k - \hat{k})|_O \equiv 0 \quad \Rightarrow \quad \dim O = 2,$$
$$(k - \hat{k})|_O \not\equiv 0 \quad \Rightarrow \quad \dim O = 5.$$

(2) *There exists an injective correspondence between isometries $i : M \to \widehat{M}$ and* 2*-dimensional reachable sets O of system* (5). *In particular, if the manifolds M and $\widehat{M}$ are isometric, then system* (5) *is not globally controllable.*

(3) *Suppose that both manifolds M and $\widehat{M}$ are simply connected. Then the correspondence between isometries $i : M \to \widehat{M}$ and* 2*-dimensional reachable sets O of system* (5) *is bijective. In particular, system* (5) *is globally controllable iff the manifolds M and $\widehat{M}$ are not isometric.*

Proof. (1) By the Stefan-Sussmann orbit theorem (see [10, 11]), the reachable set of the symmetric system (5), i.e., the orbit of the distribution Δ through any point $q \in Q$ is an immersed smooth connected submanifold of Q. Now we show that any orbit O of Δ has dimension either 2 or 5. Indeed, fix an orbit O and assume first that at some point $q \in O$ the surfaces M and $\widehat{M}$ have different Gaussian curvatures: $k(q) \neq \hat{k}(q)$. The direct computation gives

$$[X_1, X_2] = \alpha_1 X_1 + \alpha_2 X_2 + (\hat{k} - k)\frac{\partial}{\partial\theta}, \tag{6}$$

thus

$$\frac{\partial}{\partial\theta} \in \mathrm{Lie}\,(X_1, X_2). \tag{7}$$

Since

$$\begin{aligned}
[\frac{\partial}{\partial\theta}, X_1] &= -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2 + (\cdots)\,\frac{\partial}{\partial\theta},\\
[\frac{\partial}{\partial\theta}, X_2] &= -\cos\theta\,\hat{e}_1 - \sin\theta\,\hat{e}_2 + (\cdots)\,\frac{\partial}{\partial\theta},
\end{aligned}$$

then

$$\hat{e}_1, \hat{e}_2 \in \mathrm{Lie}\,(X_1, X_2). \tag{8}$$

Taking into account (3), (4), (7), and (8), we obtain

$$e_1, e_2 \in \mathrm{Lie}\,(X_1, X_2).$$

So we have

$$\mathrm{Lie}\,(X_1, X_2)(q) = \mathrm{span}\,\left(e_1, e_2, \hat{e}_1, \hat{e}_2, \frac{\partial}{\partial\theta}\right)(q) = T_q Q.$$

System (5) has the full rank at the point $q \in O$, hence $\dim O = 5$.

On the other hand, if $k(q) = \hat{k}(q)$ at all points $q \in O$, then (6) implies that the distribution Δ is integrable, thus $\dim O = 2$.

(2) Let $i : M \to \widehat{M}$ be an isometry. Its graph

$$\begin{aligned}
\Gamma = \{\, A \in Q \mid A = i_*|_x \,:\, T_x M \to T_{\hat{x}}\widehat{M},&\\
x \in M,\ \hat{x} = i(x) \in \widehat{M} \,\}&
\end{aligned}$$

is a smooth 2-dimensional submanifold of Q. We prove that Γ is an orbit of Δ. Locally, choose an orthonormal frame e_1, e_2 in M and take the corresponding orthonormal frame $\hat{e}_1 = i_* e_1$, $\hat{e}_2 = i_* e_2$ in $\widehat{M}$. Then Γ is locally defined by the equation $\theta = 0$. Taking into account that $\hat{\alpha}_1 = \alpha_1$, $\hat{\alpha}_2 = \alpha_2$, we obtain then

$$X_1|_\Gamma = e_1 + \hat{e}_1, \quad X_2|_\Gamma = e_2 + \hat{e}_2,$$

that is why the fields X_1, X_2 are tangent to Γ. In view of (6),

$$[X_1, X_2]|_\Gamma = \alpha_1 X_1 + \alpha_2 X_2.$$

It follows that Γ is an orbit of Δ. Distinct isometries $i_1 \neq i_2$ have distinct graphs $\Gamma_1 \neq \Gamma_2$, i.e., the correspondence between isometries and 2-dimensional orbits is injective.

(3) Now assume that the manifolds M, $\widehat{M}$ are simply connected. Let O be a 2-dimensional orbit of Δ. We construct an isometry $i : M \to \widehat{M}$ with the graph O. The mappings

$$\pi : O \to M \quad \text{and} \quad \hat{\pi} : O \to \widehat{M} \tag{9}$$

are local diffeomorphisms since $\pi_*(X_1) = e_1$, $\pi_*(X_2) = e_2$, $\hat{\pi}_*(X_1) = \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2$, $\hat{\pi}_*(X_2) = -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2$. Moreover, $\pi(O)$ is an orbit of the distribution $\operatorname{span}(e_1, e_2)$, i.e., $\pi(O) = M$. Analogously, $\hat{\pi}(O) = \widehat{M}$. The fact that M and $\hat{M}$ are simply connected implies that the mappings (9) are (global) diffeomorphisms even if O is not compact. Indeed, let $q \in O$; we know that any Lipschitzian curve $x(\cdot)$ on M starting from $\pi(q)$ has a unique admissible lift on O starting from q and this lift continuously depends on $x(\cdot)$. Suppose that $q' \in O$, $q' \neq q$, $\pi(q') = \pi(q)$, and $q(\cdot)$ is a path on O connecting q with q'. Clearly, any Lipschitzian path on O is admissible. Contracting the loop $\pi(q(\cdot))$ and taking the lift of the contraction we come to a contradiction with the local invertibility of $\pi|_O$. Hence $\pi|_O$ is globally invertible. The same arguments show that $\hat{\pi}|_O$ is globally invertible.

Thus we can define a diffeomorphism

$$i = \hat{\pi} \circ (\pi|_O)^{-1} : M \to \widehat{M}.$$

Since

$$\begin{aligned} i_* e_1 &= \cos\theta\,\hat{e}_1 + \sin\theta\,\hat{e}_2, \\ i_* e_2 &= -\sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_2, \end{aligned}$$

it follows that i is an isometry.

If the manifolds M and $\widehat{M}$ are not isometric, then all reachable sets of system (5) are 5-dimensional, thus open subsets of Q. But Q is connected, thus it is a single reachable set. $\square$

3 Optimal control problem

Consider a contact configuration $q_0 \in Q$ of the rolling bodies with $k(q_0) \neq \hat{k}(q_0)$. System (5) has the full rank and is thus locally controllable at q_0: for any sufficiently close configuration $q_1 \in Q$ there exists an admissible motion of the system that steers q_0 to q_1.

In the neighborhood of q_0, we seek for motions of the system optimal in the sense of minimality of length of the path $x(t)$ of the contact point in M (or, equivalently, of the path $\hat{x}(t)$ in $\widehat{M}$). This optimal control problem is defined by dynamics (5) and the cost functional

$$l = \int \sqrt{u_1^2 + u_2^2}\, dt \to \min. \tag{10}$$

Optimal trajectories are geodesics of the rank 2 sub-Riemannian structure on Q determined by the orthonormal frame X_1, X_2:

$$\Delta = \mathrm{span}\,(X_1, X_2), \quad \langle X_i, X_j \rangle_Q = \delta_{ij}, \ i, j = 1, 2. \tag{11}$$

4 Nilpotent approximation

We replace the sub-Riemannian structure (11) by its nilpotent approximation in the neighborhood of q_0 and study geodesics of this nilpotent structure. For the general construction of nilpotent approximations of control systems and sub-Riemannian structures, see [1, 2, 3].

Consider the flag of the distribution Δ:

$$\Delta \subset \Delta^2 = \Delta + [\Delta, \Delta] \subset \Delta^3 = \Delta^2 + [\Delta, \Delta^2] \subset \dots .$$

Since $k(q_0) \neq \hat{k}(q_0)$, the growth vector of Δ is

$$\dim \Delta(q) = 2, \quad \dim \Delta^2(q) = 3, \quad \dim \Delta^3(q) = 5$$

for $q \in Q$ sufficiently close to q_0, that is, the distribution Δ is regular at q_0. The degree of nonholonomy of Δ is 3.

In a neighborhood U_{q_0} of the point q_0 there exist a pair of vector fields (Y_1, Y_2) that provides a nilpotent approximation of the system (X_1, X_2) at q_0. The Lie algebra

$$L = \mathrm{Lie}\,(Y_1, Y_2)$$

is nilpotent and spanned by the vector fields

$$Y_1, \ Y_2, \ Y_3 = [Y_1, Y_2], \ Y_4 = [Y_1, Y_3], \ Y_5 = [Y_2, Y_3], \tag{12}$$

all the rest Lie brackets in L either are zero or follow from (12) by skew-symmetry. There exists an adapted coordinate mapping $\Phi : U_{q_0} \to \mathbf{R}^5$ in which Y_i, $i = 1, \dots, 5$, become polynomial vector fields. The space $\mathbf{R}^5$ can be identified with the connected simply connected Lie group G corresponding to the Lie algebra L.

The control system (5) and sub-Riemannian structure (11) are approximated in the neighborhood U_{q_0} respectively by the system

$$\dot{g} = u_1 Y_1(g) + u_2 Y_2(g), \quad g \in G, \quad u_1, \, u_2 \in \mathbf{R}, \tag{13}$$

and the sub-Riemannian structure

$$D = \mathrm{span}\,(Y_1, Y_2), \quad \langle Y_i, Y_j \rangle_G = \delta_{ij}, \ i, j = 1, 2, \tag{14}$$

in the following sense.

Let $q(t)$ and $g(t)$ be trajectories of systems (5) and (13) respectively corresponding to some control functions $u_1(t)$, $u_2(t)$, $u_1^2(t)+u_2^2(t) \equiv 1$, starting from the point $Q \ni q_0 = e \in G$.

In adapted coordinates $(g_1,\dots,g_5)$ in the neighborhood U_{q_0}, consider a dilation

$$\delta_\varepsilon \,:\, (g_1,g_2,g_3,g_4,g_5) \mapsto (\varepsilon g_1, \varepsilon g_2, \varepsilon^2 g_3, \varepsilon^3 g_4, \varepsilon^3 g_5).$$

Trajectories $g(t)$ of the nilpotent system (13) are homogeneous with respect to this dilation

$$\delta_\varepsilon g(t) = g(\varepsilon t)$$

and approximate trajectories $q(t)$ of system (5):

$$\delta_{\frac{1}{t}}(q(t) - g(t)) = O(t), \quad t \to 0.$$

There exist $\gamma > 0$ and $C > 0$ such that for all $|t| < \gamma$

$$d_G(q(t), g(t)) \leq C t^{1+1/3},$$

where d_G is the sub-Riemannian distance in G corresponding to the sub-Riemannian structure (14):

$$d_G(g_0, g_1) = \inf \left\{ \int_0^1 \sqrt{u_1^2(t) + u_2^2(t)}\, dt \right\},$$

the infimum is taken over all trajectories $g(\cdot)$ of system (13) such that $g(0) = g_0$, $g(1) = g_1$.

5 Symmetries of the nilpotent system

We seek for infinitesimal symmetries of the sub-Riemannian structure (14) and the underlying distribution D.

The distribution D can be realized on a 5-dimensional homogeneous space of the 14-dimensional exceptional simple Lie group G_2 in such a way that D is preserved by the group G_2. Hence the algebra of symmetries of D has dimension not less than 14.

To find the Lie algebra of infinitesimal symmetries of the distribution D, we choose the following model:

$$\begin{aligned}
&G = \mathbf{R}^5_{x,y,z,v,w},\\
&(x_1,y_1,z_1,v_1,w_1)\cdot(x_2,y_2,z_2,v_2,w_2) =\\
&(x_1+x_2, y_1+y_2, z_1+z_2+x_1y_2,\\
&\quad v_1+v_2+z_1y_2+x_1y_2^2/2,\\
&\quad w_1+w_2+2x_1z_2+x_1^2y_2),\\
&Y_1 = \frac{\partial}{\partial x}, \quad Y_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z} + z\frac{\partial}{\partial v} + x^2\frac{\partial}{\partial w}.
\end{aligned}$$

Theorem 2 (E. Cartan [6]) *The Lie algebra of infinitesimal symmetries of the left-invariant distribution* $D = \mathrm{span}\,(Y_1, Y_2)$ *on the five-dimensional nilpotent Lie group* G *is a 14-dimensional Lie algebra — the (unique) noncompact real form of the complex exceptional simple Lie algebra* g_2*.*

For the model in $\mathbf{R}^5$, this Lie algebra is spanned by the following vector fields:

$$
\begin{aligned}
\xi_1 &= \frac{1}{36}\left(2\frac{\partial}{\partial x} + 2y\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial v} + 4z\frac{\partial}{\partial w}\right),\\
\xi_2 &= -3\frac{\partial}{\partial y},\\
\xi_3 &= \frac{1}{12}\left(\frac{\partial}{\partial z} + y\frac{\partial}{\partial v}\right),\\
\xi_4 &= -\frac{1}{324}\frac{\partial}{\partial w},\\
\xi_5 &= \frac{1}{12}\frac{\partial}{\partial v},\\
\xi_6 &= \frac{1}{3}\left((6x^2y^2 - 12xyz + 24z^2 - 18yw)\frac{\partial}{\partial x}\right.\\
&\quad +(6xy^3 - 24zy^2 + 36yv)\frac{\partial}{\partial y}\\
&\quad +(3x^2y^3 - 12yz^2 - 9y^2w + 36zv)\frac{\partial}{\partial z}\\
&\quad +(6xy^3z - 12y^2z^2 - 3y^3w + 36v^2)\frac{\partial}{\partial v}\\
&\quad \left.+\,(2x^3y^3 - 36yzw + 16z^3 + 36vw)\frac{\partial}{\partial w}\right),\\
\xi_7 &= 3\left((-4x^2y + 4xz + 6w)\frac{\partial}{\partial x}\right.\\
&\quad +(-6xy^2 + 16yz - 12v)\frac{\partial}{\partial y}\\
&\quad +(-3x^2y^2 + 4z^2 + 6wy)\frac{\partial}{\partial z}\\
&\quad +(-6xy^2z + 8yz^2 + 3y^2w)\frac{\partial}{\partial v}\\
&\quad \left.+\,(-2x^3y^2 + 12zw)\frac{\partial}{\partial w}\right),\\
\xi_8 &= 36\left((6x^2z - 9xw)\frac{\partial}{\partial x} + (18xv - 12z^2)\frac{\partial}{\partial y}\right.\\
&\quad +(9x^2v - 9zw)\frac{\partial}{\partial z} + (18xzv - 8z^3 - 9vw)\frac{\partial}{\partial v}
\end{aligned}
$$

$$
\begin{aligned}
& +(6x^3v - 9w^2)\frac{\partial}{\partial w}\Bigg), \\
\xi_9 &= 9\left(2x^2\frac{\partial}{\partial x} + (6xy - 8z)\frac{\partial}{\partial y} + (3x^2y - 3w)\frac{\partial}{\partial z}\right. \\
&\quad \left. +(6xyz - 4z^2 - 3yw)\frac{\partial}{\partial v} + 2x^3y\frac{\partial}{\partial w}\right), \\
\xi_{10} &= \frac{1}{3}\left((xy - 4z)\frac{\partial}{\partial x} - y^2\frac{\partial}{\partial y} - (zy + 3v)\frac{\partial}{\partial z}\right. \\
&\quad \left. -3yv\frac{\partial}{\partial v} - 4z^2\frac{\partial}{\partial w}\right), \\
\xi_{11} &= -9\left(6x\frac{\partial}{\partial y} + 3x^2\frac{\partial}{\partial z} + (6xz - 3w)\frac{\partial}{\partial v}\right. \\
&\quad \left. +2x^3\frac{\partial}{\partial w}\right), \\
\xi_{12} &= \frac{1}{324}\left(6y\frac{\partial}{\partial x} + 3y^2\frac{\partial}{\partial z} + y^3\frac{\partial}{\partial v}\right. \\
&\quad \left. +(12yz - 12v)\frac{\partial}{\partial w}\right), \\
\xi_{13} &= -x\frac{\partial}{\partial x} - z\frac{\partial}{\partial z} - v\frac{\partial}{\partial v} - 2w\frac{\partial}{\partial w}, \\
\xi_{14} &= -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w}.
\end{aligned}
$$

The vector fields $\xi_1, \ldots, \xi_5$ generate left translations on the Lie group $G = \mathbf{R}^5_{x,y,z,v,w}$, and the symmetries $\xi_{11}, \ldots, \xi_{14}$ are nonzero endomorphisms of the Lie algebra L of G.

For the sub-Riemannian structure (14), the left-invariant property implies that the group G preserves this sub-Riemannian structure. On the other hand, the model in $\mathbf{R}^5$ yields that the whole algebra of infinitesimal symmetries of (14) is 6-dimensional: in addition to 5 independent left translations in G, there is one extra symmetry — a rotation having a fixed point in G.

Theorem 3 *The Lie algebra of infinitesimal symmetries of the left-invariant sub-Riemannian structure* (14) *on the five-dimensional nilpotent Lie group G is the six-dimensional Lie algebra* span$(\eta_0, \ldots, \eta_5)$ *with the following multiplication rules for the base elements:*

$$
\begin{array}{ll}
[\eta_0, \eta_1] = -\eta_2, & [\eta_0, \eta_2] = \eta_1, \\
[\eta_0, \eta_4] = -\eta_5, & [\eta_0, \eta_5] = \eta_4, \\
[\eta_1, \eta_2] = \eta_3, & \\
[\eta_1, \eta_3] = \eta_4, & [\eta_2, \eta_3] = \eta_5.
\end{array}
$$

For the model in $\mathbf{R}^5$*,*

$$\eta_0 = -\frac{1}{54}\xi_{11} - 54\xi_{12},$$
$$\eta_1 = -3\xi_1, \quad \eta_2 = \frac{1}{18}\xi_2, \quad \eta_3 = -\frac{1}{3}\xi_3,$$
$$\eta_4 = 3\xi_4, \quad \eta_5 = \frac{1}{18}\xi_5,$$

where the vector fields ξ_i *are defined in Theorem* 2*.*

The proofs of Theorems 2 and 3 are given in [9].

6 Extremals of the nilpotent sub-Riemannian problem

We study extremals of the optimal control problem (13), (10).

We use the standard replacement of length (10) by action:

$$J = \frac{1}{2}\int (u_1^2 + u_2^2)\, dt \to \min \tag{15}$$

and seek for extremals of the problem (13), (15) via Pontryagin maximum principle. The control-dependent Hamiltonian is

$$h(q,p,u) = \langle p, u_1 Y_1 + u_2 Y_2\rangle - \frac{\lambda}{2}(u_1^2 + u_2^2),$$
$$q \in Q, \ p \in T_q^* Q,$$

where λ can take values 1 (the regular case) and 0 (the abnormal case).

First we find regular extremals. The Pontryagin maximum principle yields

$$\frac{\partial h}{\partial u} = 0,$$

thus

$$u_1 = \langle p, Y_1\rangle, \quad u_2 = \langle p, Y_2\rangle.$$

We pass to the angle coordinate φ:

$$u_1 = \cos\varphi, \quad u_2 = \sin\varphi,$$

use the notation $u_i = \langle p, Y_i\rangle$, $i = 3, 4, 5$, and obtain

$$\dot\varphi = u_3, \tag{16}$$
$$\dot u_3 = u_4 \cos\varphi - u_5 \sin\varphi, \tag{17}$$
$$\dot u_4 = 0,$$
$$\dot u_5 = 0.$$

Thus u_4, u_5 are constant, and equations (16), (17) read

$$\ddot{\varphi} = -\alpha \sin(\varphi - \beta), \tag{18}$$

where $\alpha = \sqrt{u_4^2 + u_5^2}$, $\sin \beta = u_4 / \sqrt{u_4^2 + u_5^2}$, $\cos \beta = u_5 / \sqrt{u_4^2 + u_5^2}$. Equation (18) is the pendulum equation and is integrable in elliptic functions.

For abnormal extremals, we obtain the lines

$$\varphi \equiv \mathrm{const} = -\beta,$$

which are also solutions to the pendulum equation (18).

Thus all extremal controls of the nilpotent optimal control problem (13), (10) are expressed in terms of elliptic functions. This fact was already noticed by R. Brockett and L. Dai [5]. Applying the model from section 5 one obtains an explicit expression of the extremal trajectories via quadratures of elliptic functions.

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