

# On optimal rolling of a ball on a plane with twisting and without slipping

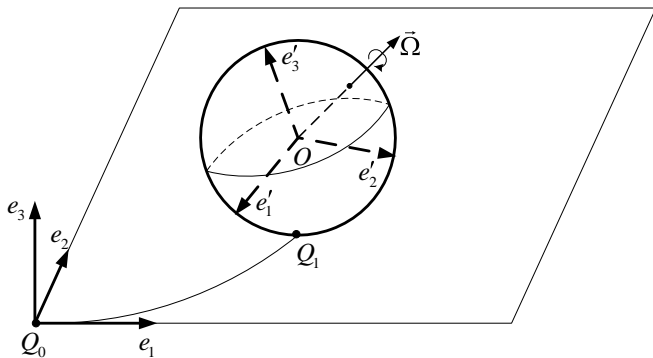
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## Problem Statement

A ball rolls on a plane with twisting but without slipping.



Stationary frame  $e_i$ , moving frame  $e'_i$ ,  
angular velocity vector  $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$

## Problem Statement

**Problem:** Roll the sphere from an initial configuration  $Q_0 = (x_0, y_0, R_0)$  to the final configuration  $Q_1 = (x_1, y_1, R_1)$  so, that the action is minimized:

$$\int_0^{t_1} \frac{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}{2} dt \rightarrow \min$$

Optimal control problem:

$$\dot{x} = u_1, \quad \dot{y} = u_2, \quad \dot{R} = R \begin{pmatrix} 0 & -u_3 & -u_1 \\ u_3 & 0 & -u_2 \\ u_1 & u_2 & 0 \end{pmatrix},$$

$$Q = (x, y, R) \in \mathbb{R}^2 \times SO(3), \quad (u_1, u_2, u_3) \in \mathbb{R}^3$$

$$Q_0 = (x_0, y_0, R_0), \quad Q_1 = (x_1, y_1, R_1)$$

$$\int_0^{t_1} \frac{u_1^2 + u_2^2 + u_3^2}{2} dt \rightarrow \min, \quad t_1 \text{ fixed.}$$

# Motivation for studying

1. Applications in mechanics and robotics;
2. The original ball-plate problem (without slipping or twisting) is not solved;
3. An important problem of sub-Riemannian geometry.

## Sub-Riemannian problem

Left-invariant vector fields on  $G = \mathbb{R}^2 \times SO(3)$ :

$$X_1 = e_1 - V_1, \quad X_2 = e_2 + V_2, \quad X_3 = V_3$$

where

$$e_1 = \frac{\partial}{\partial x}, \quad e_2 = \frac{\partial}{\partial y},$$

$$V_1 = R \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad V_2 = R \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad V_3 = R \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

define a distribution of a constant rank:

$$\Delta = \text{span}\{X_1, X_2, X_3\}$$

## Sub-Riemannian problem

The control equations:

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q) + u_3 X_3(Q)$$

describe admissible curves on  $G$ , i.e. curves whose velocity vectors lie in  $\Delta$  for any  $t$ .

The minimization of action is equivalent to the minimization of the length functional:

$$\frac{1}{2} \int_0^{t_1} (u_1^2 + u_2^2 + u_3^2) dt \rightarrow \min \iff \int_0^{t_1} \sqrt{u_1^2 + u_2^2 + u_3^2} dt \rightarrow \min$$

**Sub-Riemannian problem:** Given two points  $Q_0, Q_1 \in G$  and a distribution  $\Delta$  find an admissible curve with minimal length, which connects  $Q_0$  with  $Q_1$ .

## Controllability and existence of optimal solutions

$$\dot{Q} = \sum_{i=1}^3 u_i X_i(Q), \quad Q \in \mathbb{R}^2 \times SO(3), \quad (u_1, u_2, u_3) \in \mathbb{R}^3$$

- $\text{span}\{X_1, X_2, X_3, [X_1, X_3], [X_2, X_3]\}(Q) = T_Q G$  for any  $Q$ , and by the Rashevsky-Chow theorem system is completely controllable.
- Existence of optimal trajectories follows from Filippov's theorem.

## Parametrization of abnormal geodesics

### Theorem

*Abnormal geodesics are parametrized by the following equations*

$$x = -\Omega_2 t, \quad y = \Omega_1 t, \quad R = e^{\tilde{\Omega} t},$$

where

$$\tilde{\Omega} = \begin{pmatrix} 0 & 0 & \Omega_2 \\ 0 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

*and are optimal on any segment  $[0, t_1]$ ,  $t_1 > 0$ .*

# Rolling along a line without twisting or slipping

Abnormal trajectory

## Normal Hamiltonian system of Pontryagin's Maximum Principle (PMP)

$$\begin{aligned}\dot{x} &= \Omega_2, & \dot{\Omega}_1 &= \omega_2 \Omega_3, \\ \dot{y} &= -\Omega_1, & \dot{\Omega}_2 &= -\omega_1 \Omega_3, \\ \dot{R} &= R\tilde{\Omega}, & \dot{\Omega}_3 &= \omega_1 \Omega_2 - \omega_2 \Omega_1 \\ & & \omega_1, \omega_2 &= \text{const.}\end{aligned}$$

$$\frac{\Omega_1^2 + \Omega_2^2 + \Omega_3^2}{2} = \text{const}, \quad (\omega_1, \omega_2) \in \mathbb{R}^2$$

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ -\Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}$$

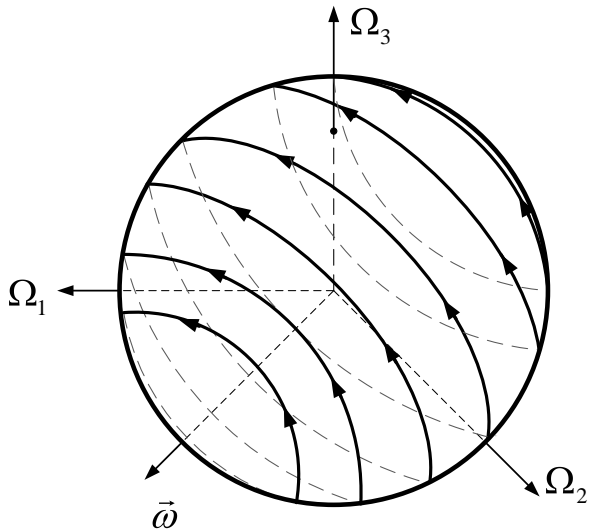
## Normal Hamiltonian system of PMP

Equations for the angular velocity vector can be written in an equivalent way:

$$\dot{\vec{\Omega}} = \vec{\omega} \times \vec{\Omega}$$

where  $\vec{\omega} = (\omega_1, \omega_2, 0)$ ,  $\vec{\Omega} = (\Omega_1, \Omega_2, \Omega_3)$  This equation implies that the velocity vector  $\vec{\Omega}$  rotates uniformly around a static horizontal vector  $\vec{\omega}$ .

Uniform rotation of  $\vec{\Omega}$  around  $\vec{\omega}$



## Rolling along a line

If  $\vec{\Omega} = \lambda \vec{\omega}$  for some  $\lambda \neq 0$  and  $\vec{\omega} \neq 0$ , then by integrating the Hamiltonian system of PMP, we get the equations of abnormal geodesics.

If  $\vec{\omega} \equiv 0$ , then  $\vec{\Omega} = \text{const}$  and the normal geodesics are parametrized by equations:

$$x = -\Omega_2 t, \quad y = \Omega_1 t, \quad R = e^{\tilde{\Omega} t},$$

where

$$\tilde{\Omega} = \begin{pmatrix} 0 & -\Omega_3 & \Omega_2 \\ -\Omega_3 & 0 & -\Omega_1 \\ -\Omega_2 & \Omega_1 & 0 \end{pmatrix}.$$

Thus the ball rolls along a line with twisting.

## Rolling along a line with twisting

Normal trajectory (a special case):

# Rotational symmetry of the Hamiltonian system

## Theorem

Rotation  $\{\Phi^\beta, \beta \in S^1\}$  defined in the following way:

$$\Phi^\beta : \{\tilde{\Omega}_s, \tilde{\omega}_s, Q_s | s \in [0, t]\} \mapsto \{\tilde{\Omega}_s^\beta, \tilde{\omega}_s^\beta, Q_s^\beta | s \in [0, t]\},$$

$$\tilde{\Omega}_s^\beta = e^{\beta A_3} \tilde{\Omega}_s e^{-\beta A_3}, \quad \tilde{\omega}_s^\beta = e^{\beta A_3} \tilde{\omega}_s e^{-\beta A_3},$$

$$\begin{pmatrix} x_s^\beta \\ y_s^\beta \end{pmatrix} = \begin{pmatrix} \cos \beta & -\sin \beta \\ \sin \beta & \cos \beta \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix},$$

$$R_s^\beta = e^{\beta A_3} R_s e^{-\beta A_3}$$

is a symmetry of the Hamiltonian system of PMP.

## Normal Hamiltonian system of PMP

Thus we can assume that  $\vec{\omega} = (\omega, 0, 0)$ , then the extremal trajectories are described by:

$$x = -\omega^{-1}(\Omega_2^0 \sin \omega t + \Omega_3^0 \cos \omega t - \Omega_3^0), \quad y = \Omega_1^0 t,$$

$$R(t) = e^{t(\tilde{\omega} + \tilde{\Omega}_0)} e^{-t\tilde{\omega}}.$$

The ball rolls along a sinusoid, and the rotation matrix is a composition of two rotations around vector  $-\vec{\omega}$  and  $\vec{\omega} + \vec{\Omega}_0$ .

We can use  $\Phi^\beta$  to obtain equations for an extremal trajectory with an arbitrary  $\vec{\omega}$ .

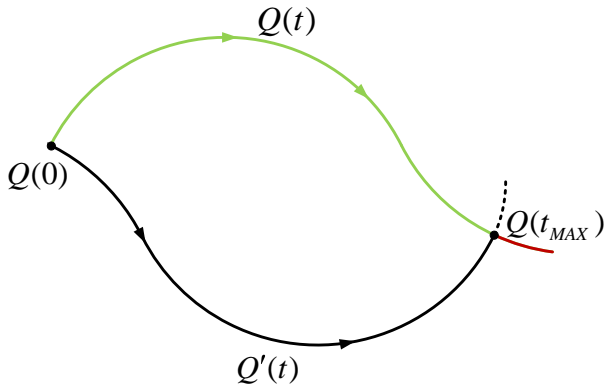
## Rolling along a sinusoid

Normal trajectory (the general case):

## Maxwell points and the cut time

Cut time is the instant of time when an extremal trajectory loses its optimality.

One can estimate it from above by finding Maxwell points, i.e. points where two distinct geodesics of the same length meet one another. After a Maxwell point a geodesic is not globally optimal.



# Maxwell points and the cut time

## The exponential mapping

$$\text{Exp} : T_{Q_0}^* G \times \mathbb{R}_+ \rightarrow G,$$

$$\text{Exp}(\lambda, t) = Q_t = (x_t, y_t, R_t),$$

$$\lambda = (\vec{\Omega}_0, \vec{\omega}) \in S^2 \times \mathbb{R}^2, \quad t \in \mathbb{R}_+, \quad Q_t \in G.$$

$$\text{MAX} = \{(\lambda, t) : \exists \tilde{\lambda} \neq \lambda, \text{Exp}(\lambda, t) = \text{Exp}(\tilde{\lambda}, t)\}$$

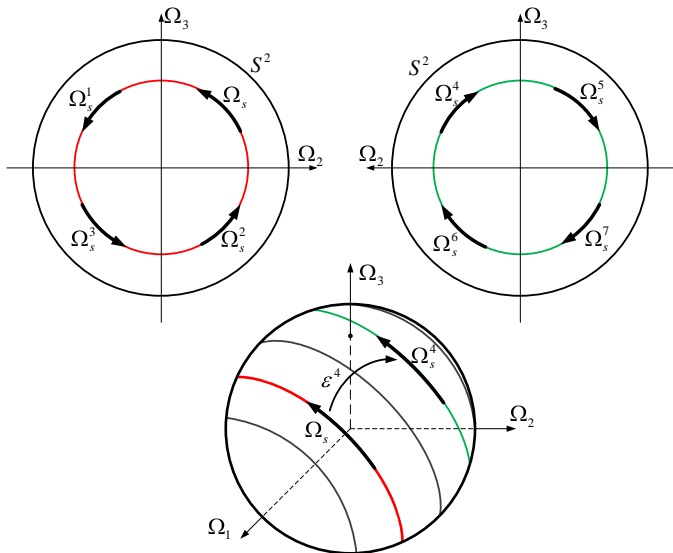
# Group of discrete symmetries of the exponential mapping

Multiplication table in the group of discrete symmetries

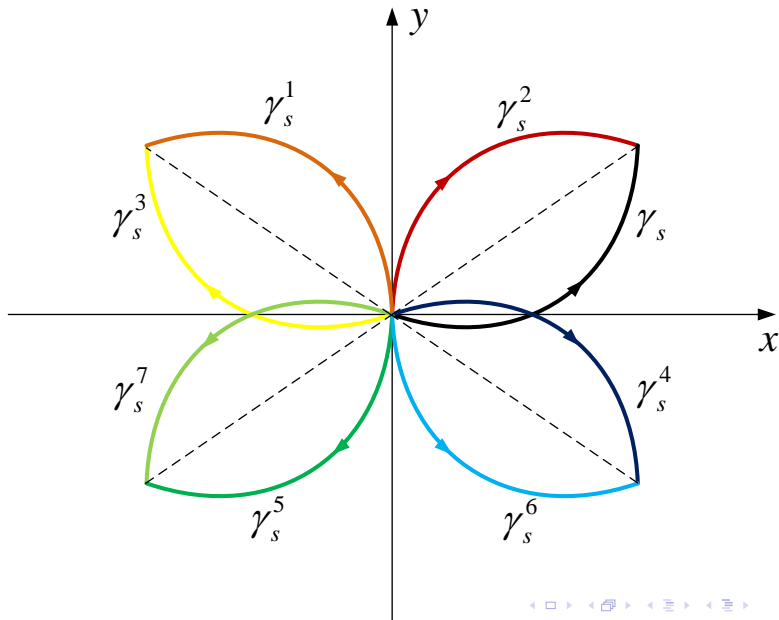
$\langle \varepsilon^1, \dots, \varepsilon^7 \rangle = \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ :

	$\varepsilon^1$	$\varepsilon^2$	$\varepsilon^3$	$\varepsilon^4$	$\varepsilon^5$	$\varepsilon^6$	$\varepsilon^7$
$\varepsilon^1$	ld	$\varepsilon^3$	$\varepsilon^2$	$\varepsilon^5$	$\varepsilon^4$	$\varepsilon^7$	$\varepsilon^6$
$\varepsilon^2$		ld	$\varepsilon^1$	$\varepsilon^6$	$\varepsilon^7$	$\varepsilon^4$	$\varepsilon^5$
$\varepsilon^3$			ld	$\varepsilon^7$	$\varepsilon^6$	$\varepsilon^5$	$\varepsilon^4$
$\varepsilon^4$				ld	$\varepsilon^1$	$\varepsilon^2$	$\varepsilon^3$
$\varepsilon^5$					ld	$\varepsilon^3$	$\varepsilon^2$
$\varepsilon^6$						ld	$\varepsilon^1$
$\varepsilon^7$							ld

# Reflections in the preimage of the exponential mapping



## Reflections of sinusoids in the plane



## Representing rotations as quaternions

Let  $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid (q_0, \dots, q_3) \in \mathbb{R}^4\}$  be the algebra of quaternions. Any rotation  $R$  of a three-dimensional space can be represented as a pair of unit quaternions  $\pm q$ .

Suppose that  $R$  is a rotation around a unit vector  $\vec{a} = (a_1, a_2, a_3)$  on an angle  $\theta$ . Then the corresponding quaternion  $q$  is given by the formula:

$$q = \cos \frac{\theta}{2} + (a_1 i + a_2 j + a_3 k) \sin \frac{\theta}{2}$$

Thus we can represent both matrix exponentials of  $R$  as quaternions:

$$e^{t(\vec{\omega} + \vec{\Omega}_0)} = \cos \frac{|\vec{\omega} + \vec{\Omega}_0|t}{2} + \frac{(\omega + \Omega_1^0)i + \Omega_2^0j + \Omega_3^0k}{|\vec{\omega} + \vec{\Omega}_0|} \sin \frac{|\vec{\omega} + \vec{\Omega}_0|t}{2}$$

$$e^{-t\vec{\omega}} = \cos \frac{\omega t}{2} - i \sin \frac{\omega t}{2}$$

## Fixed points of symmetries in the image of the exponential mapping

Suppose that  $q = q_0 + iq_1 + jq_2 + kq_3$  is a quaternion which corresponds to the orientation of the ball in  $\mathbb{R}^3$ .

### Theorem

$$\begin{aligned} \Phi^\beta \circ \varepsilon^1(Q_t) = Q_t &\iff \begin{cases} q_3 = 0, \\ q = \pm k; \end{cases} \\ \Phi^\beta \circ \varepsilon^2(Q_t) = Q_t &\iff \begin{cases} q_1x + q_2y = 0, \\ q = \pm(\cos \chi i + \sin \chi j); \end{cases} \\ \Phi^\beta \circ \varepsilon^3(Q_t) = Q_t &\iff \begin{cases} q_1x + q_2y = 0, \\ q_3 = 0; \end{cases} \quad \text{or} \quad \begin{cases} q_1y - q_2x = 0, \\ q_0 = 0; \end{cases} \end{aligned}$$

where:

$$\cos \chi = \frac{x}{\sqrt{x^2 + y^2}}, \quad \sin \chi = \frac{y}{\sqrt{x^2 + y^2}}.$$

## Maxwell time $t_{max}^1$

### Proposition

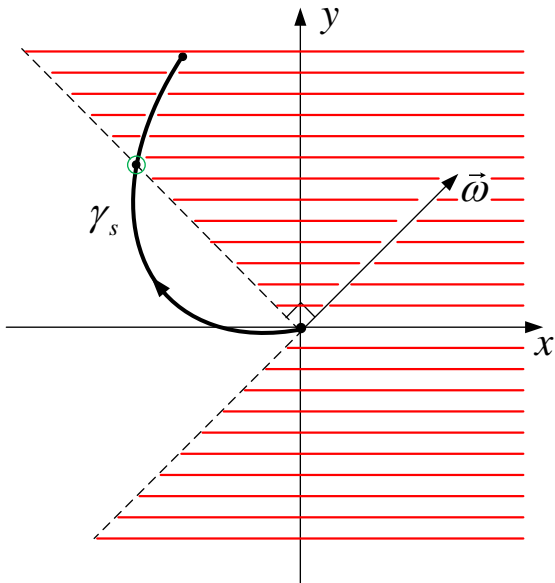
*The first positive value:*

$$t_{max}^1 = \frac{\pi(2k - 1) - 2\psi}{\omega}, \quad k \in \mathbb{Z}$$

*is a Maxwell time for all extremal trajectories whose projections on the plane are not lines.  $\psi$  is the angle between the projection of  $\vec{\Omega}_0$  on the plane  $\Omega_2 O \Omega_3$  and the axes  $O \Omega_2$*

**Geometric meaning:** At the instant of time  $t_{max}^1$  the trajectory of the contact point intersects the line perpendicular to projection of  $\vec{\omega}$  and containing the origin  $(0,0)$ .

# The geometric meaning of $t_{max}^1$



# Maxwell time $t_{max}^2$

## Proposition

The value

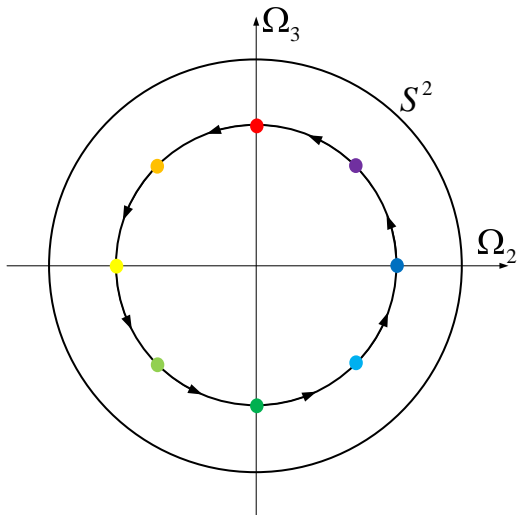
$$t_{max}^2 = \frac{2\pi}{|\vec{\omega} + \vec{\Omega}|}$$

is a Maxwell time for all extremal trajectories whose projections on the plane are not lines.

## Theorem

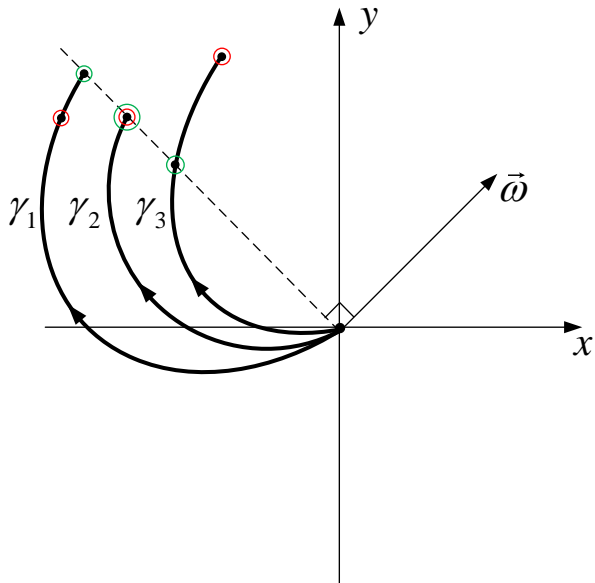
$$t_{cut} \leq \min\{t_{max}^1, t_{max}^2\}$$

# Properties of $t_{max}^2$



# Properties of $t_{max}^2$

$$t_{\max}^1 \geq t_{\max}^2 ?$$



## Which Maxwell time comes first?

### Theorem

Suppose that

$$\underline{\omega} = \frac{(\pi - \psi')^2}{\psi'(2\pi - \psi')} \left( \Omega_1^0 + \sqrt{(\Omega_1^0)^2 + \frac{\psi'(2\pi - \psi')}{(\pi - \psi')^2}} \right),$$

where

$$\psi' = \psi + \frac{\pi}{2} \pmod{\pi}$$

then:

- $t_{max}^1 < t_{max}^2 \iff \omega > \underline{\omega};$
- $t_{max}^1 = t_{max}^2 \iff \omega = \underline{\omega};$
- $t_{max}^1 > t_{max}^2 \iff \omega < \underline{\omega}.$

## How the ball can be reoriented?

**Question:** How we can roll the ball back with a prescribed orientation?

$$(x, y)(0) = (x, y)(t_1)$$

The ball can come back only if  $\vec{\Omega} \perp \vec{\omega}$ . If it's the case, then the trajectory of the contact point is a line segment.

## Comparison of the Maxwell times for the reorientation problem

The ball returns back to the origin at the time  $t_{max}^1$ . Thus  $\omega > \underline{\omega}$  must hold, where

$$\underline{\omega} = \frac{\pi - \psi'}{\sqrt{\psi'(2\pi - \psi')}},$$

Consider two regions:

$$D_+ = \{(\omega, \psi) \in \mathbb{R}^2 : t_{max}^1 < t_{max}^2, \psi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)\},$$

$$D_- = \{(\omega, \psi) \in \mathbb{R}^2 : t_{max}^1 < t_{max}^2, \psi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)\},$$

# Solution of the reorientation problem

## Theorem

If  $\vec{\omega} = (\omega, 0, 0)$ ,  $\Omega_1^0 = 0$ . Then:

- $q_2(t_{max}^1) = 0$
- The exponential mapping decomposes into diffeomorphisms  $Exp : D_+ \rightarrow Q_+$  and  $Exp : D_- \rightarrow Q_-$ , where

$$Q_+ = \{q \in \mathbb{H} : |q| = 1, q_1 > 0, q_3 > 0\},$$

$$Q_- = \{q \in \mathbb{H} : |q| = 1, q_1 > 0, q_3 < 0\}.$$

## Idea of the proof

We represent the exponential mapping as a composition of simple mappings  $f_i$  and use the following theorem:

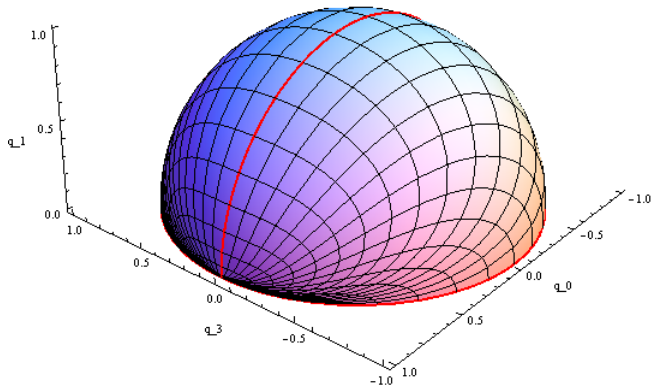
### Theorem

Let  $f : M \rightarrow N$  be a smooth mapping,  $\dim M = \dim N$ . If:

1.  $f$  is non-degenerate,
2.  $f$  is proper,
3.  $M, N$  are connected,
4.  $N$  is simply connected,

then  $f$  is a diffeomorphism.

## Image of the exponential mapping



The set  $Q_+ \cup Q_-$  of all possible orientations.

# Results

- The problem of optimal rolling on a plane with twisting is considered;
- A full parametrization of extremal curves is obtained;
- Motions of the ball are studied;
- Estimates for the cut time are obtained;
- The reorientation problem is solved.

Thank you for your attention