Sub-Riemannian problems on SO(3) and almost Riemannian geometry of the two-sphere

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Sub-Riemannian structure

Defintion: Sub-Riemannian manifold is a triple (M, Δ, g), where M is a smooth manifold, Δ is a smooth of the vector distribution and g is a Riemannian metric on this distribution.

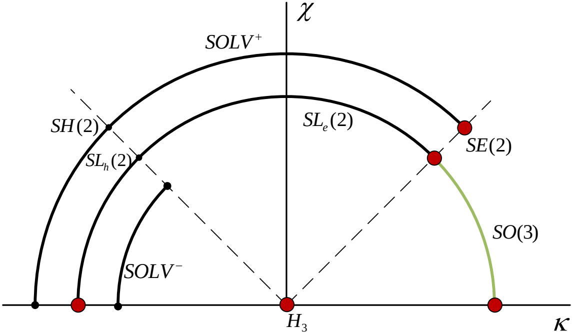
Definition: A Lipshitiz curve on M that is almost everywhere tangent to Δ is called an admissible (horizontal) curve.

Sub-Riemannian problem: for two given points $q_0, q_1 \in M$ find an admissible curve of minimal length that connects q_0 and q_1.

We're going to assume that:

1. $M = SO(3)$,
2. $\Delta = \mathrm{span}\{f_1, f_2\}$, where f_1, f_2 is a pair of left invariant vector fields
3. $g(f_i, f_j) = \delta_{ij},\ i = 1, 2.$

Classification of sub-Riemannian structures on 3D Lie groups



Red points indicate sub-Riemannian structures, where minimal length curves are completely studied.

The optimal control problem

The standard basis of $so(3)$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad A_1 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Assume that $\kappa + \chi = 1$ and $a^2 = 2\chi$. Then:

$$\dot{R} = R(u_1 A_2 + u_2 \sqrt{1-a^2} A_1) = R(u_1 f_1 + u_2 f_2) = R\Omega,$$

$$R \in SO(3), \quad (u_1, u_2) \in \mathbb{R}^2, \quad a \in (0,1),$$

$$R(0) = \mathrm{Id}, \quad R(t_1) = R_1,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \to \min .$$

Geodesic equations

After applying some first order necessary conditions, we get the following geodesics equations

$$\begin{aligned} \dot{R} &= R(p_1 f_1 + p_2 f_2) = R\Omega \\ \dot{p}_1 &= p_0 p_2 \\ \dot{p}_2 &= -p_0 p_1 \\ \dot{p}_0 &= a^2 p_1 p_2 \end{aligned}$$

Lax representation of the vertical subsystem

$$\dot{P} = [P, \Omega]$$

where $P = p_2 A_1 + \sqrt{1 - a^2} p_1 A_2 + p_0 A_3$

Integration of the vertical subsystem

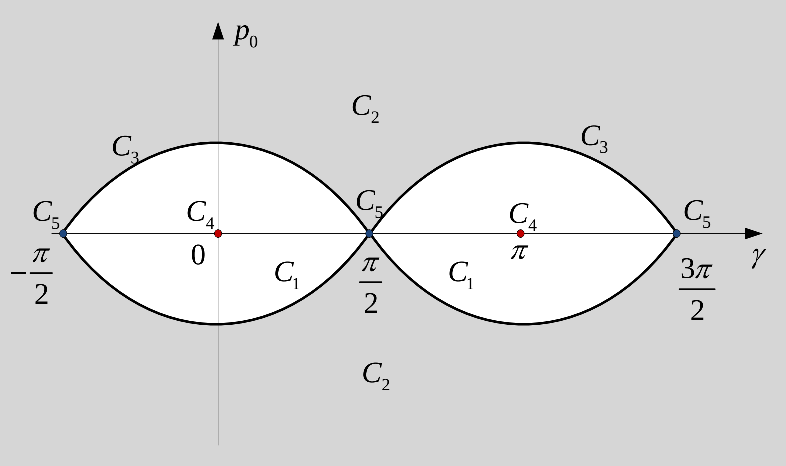
Substitution

$$p_1 = \cos\gamma, \qquad p_2 = -\sin\gamma;$$

reduces vertical subsystem to the equations of a mathematical pendulum

$$\begin{aligned} \dot\gamma &= p_0, \\ \dot p_0 &= -\frac{a^2}{2}\sin 2\gamma \end{aligned}$$

Decomposition in $T^*_{Id}G$



The equations of the mathematical pendulum can be integrated in terms of the Jacobi elliptic functions $\mathrm{sn}, \mathrm{cn}, \mathrm{dn}$.

Integration of the horizontal subsystem

Parametrisation of the geodesics on SO(3) can be represented as follows

$$R = e^{-\phi_1(0)A_3} e^{-\phi_2(0)A_1} e^{\phi_3(t)A_3} e^{\phi_2(t)A_1} e^{\phi_1(t)A_3}.$$

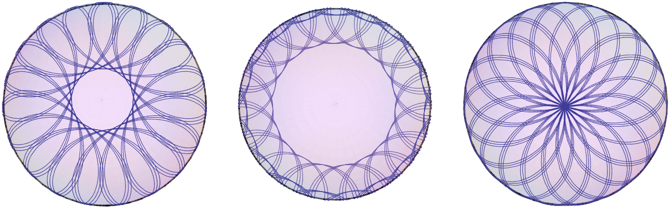
A classical approach is to get expressions of $\cos\phi_1$, $\sin\phi_1$, $\cos\phi_2$, $\sin\phi_2$ in terms of p_i

$$\cos\phi_2 = \frac{p_0}{|p|}, \qquad \sin\phi_2 = \sqrt{\frac{|p|^2 - p_0^2}{|p|^2}},$$

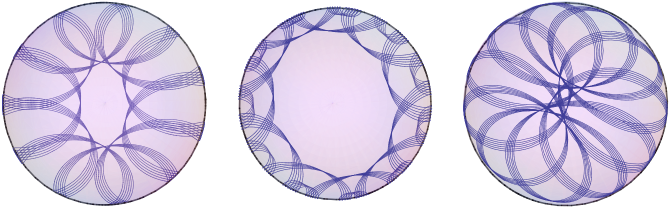
$$\cos\phi_1 = \frac{p_1\sqrt{1-a^2}}{\sqrt{|p|^2 - p_0^2}}, \qquad \sin\phi_1 = \frac{p_2}{\sqrt{|p|^2 - p_0^2}}$$

and then solve the differential equations for ϕ_3.

Projections of geodesics on a sphere



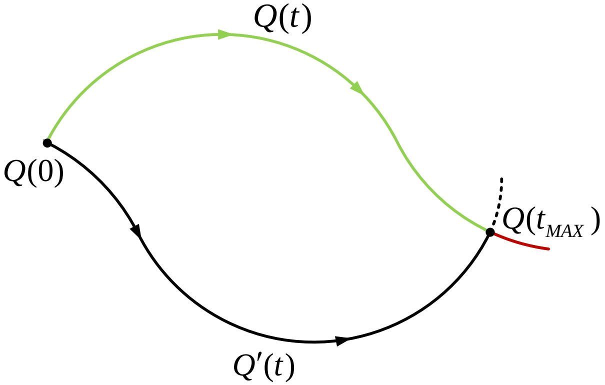
$a = 0$



$a \neq 0$

Maxwell points and the cut time

One can estimate the cut time by finding Maxwell points, i.e. points where two distinct geodesics of the same length meet one another. After a Maxwell point a geodesic is not globally optimal.



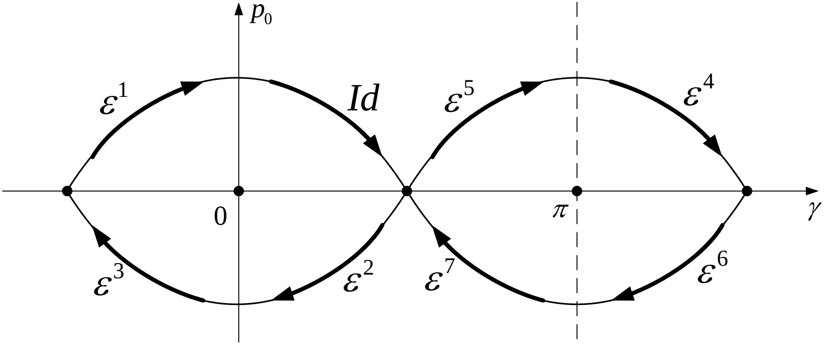
SO(3) and quaternions

Let $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 \mid (q_0, \dots, q_3) \in \mathbb{R}^4\}$ be the algebra of quaternions. Any rotation R of a three-dimensional space can be represented as a pair of unit quaternions $\pm q$.

Suppose that R is a rotation around a unit vector $\vec{a} = (a_1, a_2, a_3)$ by an angle θ. Then the corresponding quaternion q is given by the formula:

$$q = \cos\frac{\theta}{2} + (a_1 i + a_2 j + a_3 k)\sin\frac{\theta}{2}$$

Reflections in the preimage



Discrete symmetries

Suppose, that q is the corresponding unit length quaternion for the matrix $R \in \mathrm{SO}(3)$, and ω is an imaginary quaternion corresponding to $\Omega \in so(3)$.

$$\begin{array}{ll}
\varepsilon^1 : \omega_s \mapsto i\omega_{t-s}i & \varepsilon^1 : q_s \mapsto -iq_t^{-1}q_{t-s}i \\
\varepsilon^2 : \omega_s \mapsto k\omega_{t-s}k & \varepsilon^2 : q_s \mapsto -iq_t^{-1}q_{t-s}i \\
\varepsilon^3 : \omega_s \mapsto -j\omega_t j & \varepsilon^3 : q_s \mapsto -jq_s j \\
\varepsilon^4 : \omega_s \mapsto -k\omega_t k & \varepsilon^4 : q_s \mapsto -kq_s k \\
\varepsilon^5 : \omega_s \mapsto j\omega_{t-s}j & \varepsilon^5 : q_s \mapsto -jq_t^{-1}q_{t-s}j \\
\varepsilon^6 : \omega_s \mapsto -\omega_{t-s} & \varepsilon^6 : q_s \mapsto q_t^{-1}q_{t-s} \\
\varepsilon^7 : \omega_s \mapsto -i\omega_t i & \varepsilon^7 : q_s \mapsto -iq_s i
\end{array}$$

Fixed points

For SO(3):

$$\varepsilon^i(q_t) = \pm q_t,$$

These equations are equivalent to

$$q_i = 0,$$

where $i = 0, 1, 2, 3$
Unfortunately this equations are very hard to solve.

Induced problem on the two-sphere

$$\dot{\vec{x}} = \vec{x} \times \vec{\omega}, \qquad \vec{x}, \vec{\omega} \in \mathbb{R}^3, \tag{1}$$

$$|\vec{x}| = 1, \quad \vec{\omega} = u_2 \sqrt{1 - a^2} e_1 + u_1 e_2,$$

$$\vec{x}(0) = \vec{x}_0, \qquad \vec{x}(T) = \vec{x}_T,$$

$$\int_0^T \frac{u_1^2 + u_2^2}{2} dt \to \min, \quad T \text{ is fixed.}$$

Solutions of (1) are given by

$$\vec{x}_t = R_t^{-1} \vec{x}_0,$$

where $R \in SO(3)$ satisfies

$$\dot{R} = R\Omega = R(u_2 \sqrt{1 - a^2} A_1 + u_1 A_2).$$

The lifted problem

$$\dot{R} = R\Omega = R(u_2\sqrt{1-a^2}A_1 + u_1 A_2),$$

$$R \in SO(3), \qquad \Omega \in so(3),$$

$$R(0) = e^{\beta X_0}, \qquad R(T) = R_T e^{\beta X_0},$$

$$\int_0^T \frac{u_1^2 + u_2^2}{2}\,dt \to \min .$$

where $e^{\beta X_0}$ — is the rotation matrix around $\vec{x}_0$ on the angle β.

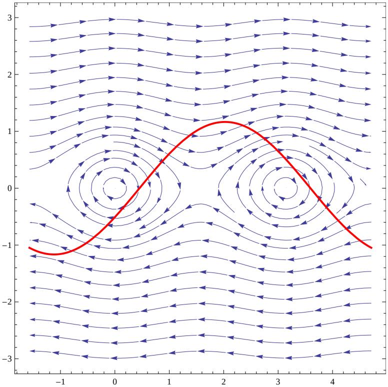
The lifted problem is almost the same as the sub-Riemannian problem, but now one has to connect two circles instead of two points.

Transversality conditions

One can proof that, if the transversality conditions are satisfied in the source, then they are automatically satisfied in the target. So:

$$\langle p_0, T\left(e^{\beta X_0}\right)\rangle = 0 \iff -\sin\gamma_0 x_0 + \cos\gamma_0 y_0\sqrt{1-a^2} + p_0(0)z_0 = 0$$

where $X_0 = xA_1 + yA_2 + zA_3$.

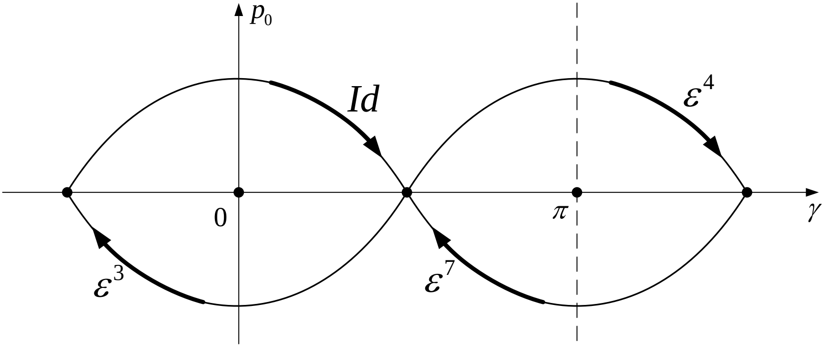


Symmetries of the problem

Consider the symmetries in the preimage that do not inverse time

$$\vec{\omega}_s \mapsto I_j \vec{\omega}_s$$

where $I_j = e^{\pi A_j}$.



Symmetries of the problem

In the image corresponding symmetries have the following form

$$\vec{x}_s \mapsto \pm I_j \vec{x}_s$$

The sign is determined by the condition, that the symmetry should preserve the initial point.

Theorem

The Maxwell sets for point x_0 are great circles that are obtained by intersecting the coordinate planes with the sphere and which contain x_0. If a geodesic crosses a Maxwell set, it's not optimal anymore.

Cut time bounds for some points

Suppose that the end of $\vec{x}_0$ lies on the equator. Then the Maxwell set is a circle given by $z_t = 0$. This is equivalent to

$$\sin\phi_3 = 0$$

From this we obtain the following result

Proposition

If $\vec{x}_0$ is a horizontal vector, then we obtain the following estimates on the cut time

1. *In C_1: $t_{cut} \leq \frac{2K(k)}{a}$;*
2. *In C_2: $t_{cut} \leq \frac{2kK(k)}{a}$;*
3. *In C_3: $t_{cut} \leq \frac{\pi}{\sqrt{1-a^2}}$;*
4. *In C_4: $t_{cut} = \pi$;*
5. *In C_5: $t_{cut} = \frac{\pi}{\sqrt{1-a^2}}$.*

Cut time bounds for the sub-Riemannian problem on SO(3)

Consequence

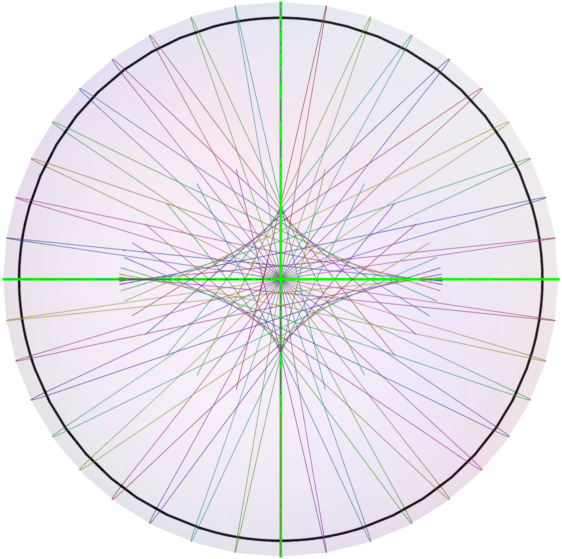
The following estimates for the cut time in the sub-Riemannian problem on SO(3) are true

1. *In* C_1: $t_{cut} \leq \frac{2K(k)}{a} + \pi$;
2. *In* C_2: $t_{cut} \leq \frac{2kK(k)}{a} + \pi$;
3. *In* C_3: $t_{cut} \leq \frac{\pi}{\sqrt{1-a^2}} + \pi$;
4. *In* C_4: $t_{cut} = 2\pi$;
5. *In* C_5: $t_{cut} = \frac{\pi}{\sqrt{1-a^2}} + \pi$.

Future work

- obtain optimal synthesis for the initial points $\vec{x}_0 = \pm e_i$, $i = 1, 2, 3$;
- study closed geodesics on SO(3) and S^2;
- find better upper-bounds on the cut time in the SR-problem;
- obtain estimates for the cut-time, when $\vec{x}_0$ is close to the singular set;
- obtain optimal synthesis in the SR-problem.

Caustic deformation



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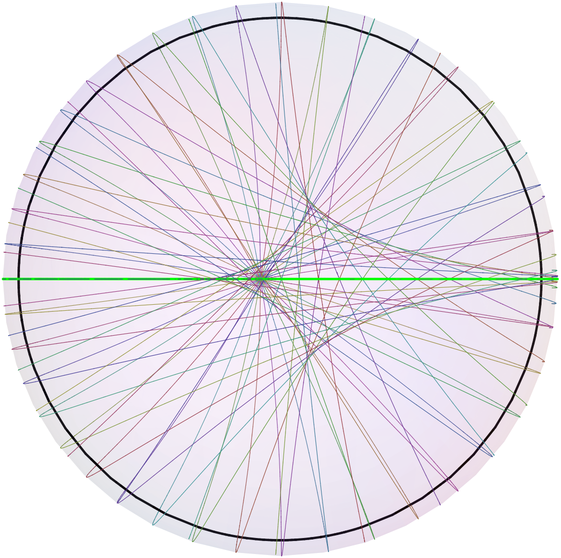

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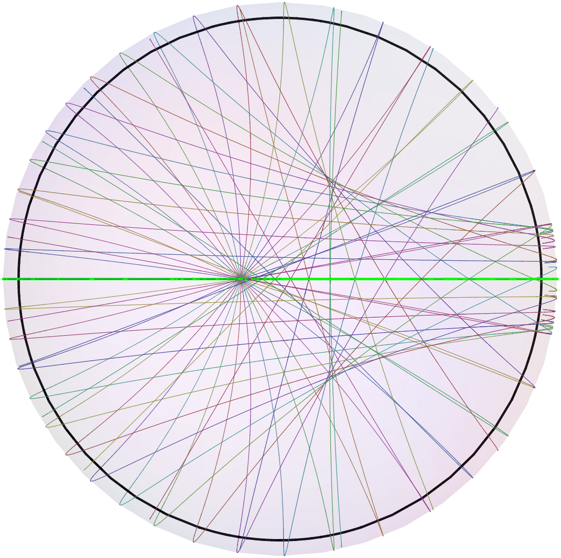

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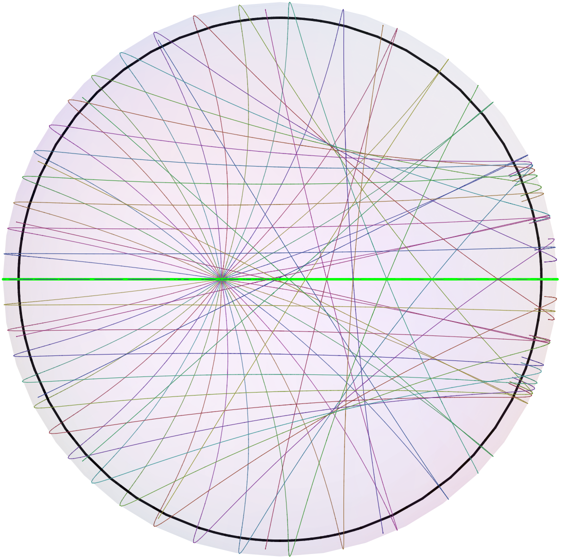

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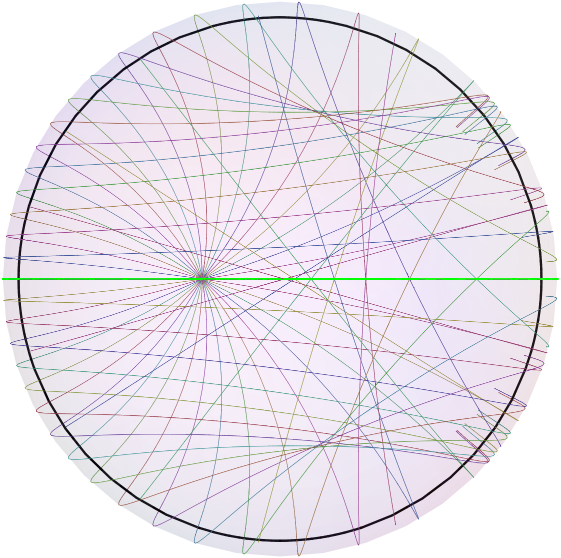

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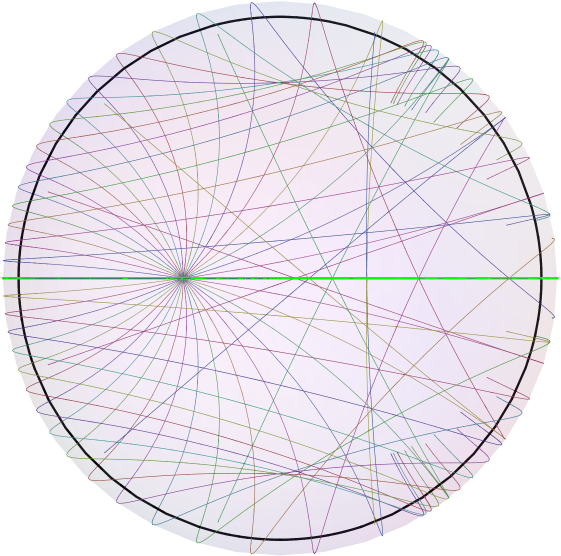

Thank you for your attention