

# Euler's elastic problem and Sub-Riemannian problem on the Engel group

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## Part 1:

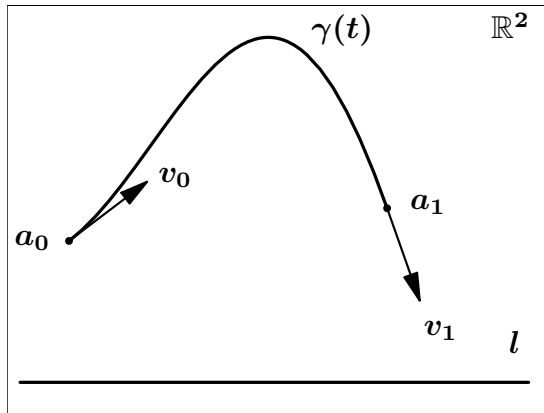
### Geometric formulation of Euler's elastic problem

Given:

$$l > 0, \quad a_0, a_1 \in \mathbb{R}^2, \\ v_0 \in T_{a_0} \mathbb{R}^2, \quad v_1 \in T_{a_1} \mathbb{R}^2, \\ |v_0| = |v_1| = 1.$$

Find:

$$\gamma(t), \quad t \in [0, t_1], \\ \gamma(0) = a_0, \quad \gamma(t_1) = a_1, \text{ s. t.} \\ \dot{\gamma}(0) = v_0, \quad \dot{\gamma}(t_1) = v_1, \\ \text{since } |\dot{\gamma}(t)| \equiv 1 \text{ then } t_1 = l \\ \int_{\gamma} k^2 dt \rightarrow \min.$$



# Coordinates in $\mathbb{R}^2 \times S^1$

Given:

$$a_0 = (x_0, y_0),$$

$$a_1 = (x_1, y_1),$$

$$\theta_0, \theta_1 \in S^1$$

$$t_1 > 0.$$

Find:

$$\gamma(t) = (x(t), y(t)),$$

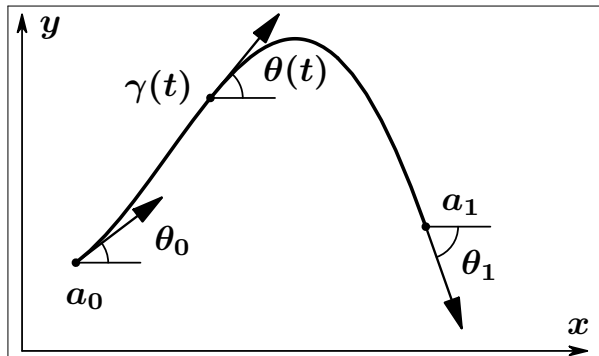
$$t \in [0, t_1], \text{ s. t.}$$

$$\gamma(0) = a_0, \gamma(t_1) = a_1,$$

$$\dot{\gamma}(0) = (\cos \theta_0, \sin \theta_0),$$

$$\dot{\gamma}(t_1) = (\cos \theta_1, \sin \theta_1),$$

$$k = \sqrt{\ddot{x}(t)^2 + \ddot{y}(t)^2} = |\dot{\theta}|.$$



## Optimal control problem statement

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} \cos \theta \\ \sin \theta \\ 0 \end{pmatrix} + u \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$q = (x, y, \theta) \in \mathbb{R}_{x,y}^2 \times \mathcal{S}_{\theta}^1, \quad u \in \mathbb{R}.$$

$$q(0) = q_0 = (x_0, y_0, \theta_0)^T, \quad q(t_1) = q_1 = (x_1, y_1, \theta_1)^T,$$

$$\int_0^{t_1} \frac{k^2}{2} dt = \int_0^{t_1} \frac{u^2}{2} dt \rightarrow \min.$$

# Normalization of conditions of the problem

- Parallel translations in  $\mathbb{R}^2 \Rightarrow (x_0, y_0) = (0, 0)$
- Rotations in  $\mathbb{R}^2 \Rightarrow \theta_0 = 0$
- Dilations in  $\mathbb{R}^2 \Rightarrow t_1 = 1$

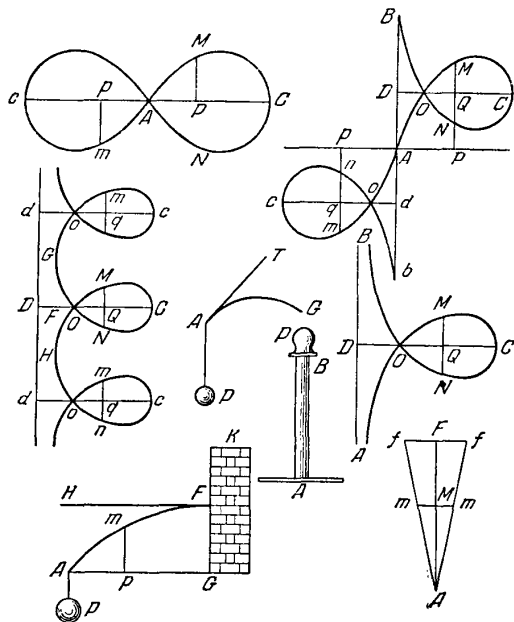
# Overview

- Parameterization of extremal curves by elliptic Jacobi's functions.
- Software for numerical computation of globally optimal curves for given boundary conditions.
- Software for construction of the cut locus.
- Sequential and parallel algorithm and software to obtain an approximate solution of system of algebraic equations.

# History of Euler's elastic problem

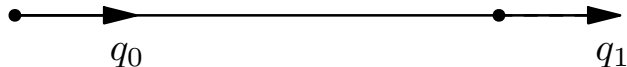
- 1691 James (Jacob) Bernoulli (Rectangular elastica)
- 1742 Daniel Bernoulli (Proposal of the variational problem)
- 1744 Leonhard Euler (Reduction to quadratures and Types of solutions)
- 1880 L.Saalchütz (Explicit parametrization of Euler elasticae by Jacobi's functions)
- 1906 Max Born (Equation of pendulum and Numeric plots of elasticae)
- 1993 Velimir Jurdjevic (Euler elasticae in the ball-plate problem)
- 1993 Roger Brockett and L. Dai (Euler elasticae in the nilpotent sub-Riemannian problem with the growth vector (2,3,5))
- 2008 Yuri Sachkov (Bounds on cut and conjugate time, reduction to systems of algebraic equations)

# Euler's sketches



# Pontryagin's maximum principle : Abnormal extremal trajectories

$$\begin{aligned}x &= t, \\y &= 0, \\ \theta &= 0\end{aligned}$$



# Normal Hamiltonian system

$$\dot{x} = \cos \theta,$$

$$\dot{y} = \sin \theta,$$

$$\dot{\theta} = c,$$

$$\dot{c} = -r \sin(\theta - \gamma),$$

$$\dot{r} = 0,$$

$$\dot{\gamma} = 0$$

$$c \in \mathbb{R},$$

$$r \in \mathbb{R}^+,$$

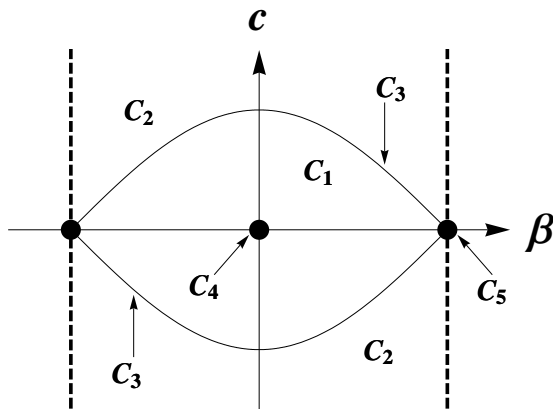
$$\gamma \in S^1.$$

$$E = \frac{\dot{\theta}^2}{2} - r \cos(\theta - \gamma) \equiv \text{const} \in [-r, +\infty).$$

# Stratification of phase cylinder of pendulum

$$C = \{\lambda = (c, \beta, r) \mid \beta = \theta - \gamma, \beta \in S^1, c \in \mathbb{R}, r \in \mathbb{R}^+\}.$$

$$C = \cup_{i=1}^5 C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j.$$



# Parametrization of extremal trajectories

$$x = x(\lambda, t),$$

$$y = y(\lambda, t),$$

$$\theta = \theta(\lambda, t)$$

by elliptic Jacobi's functions  $\text{sn}$ ,  $\text{cn}$ ,  $\text{dn}$ ,  $E$ .

# New parametrization of extremal trajectories

$$\lambda \in C_1 \Rightarrow$$

$$\sin \frac{\theta_t}{2} = k \operatorname{dn}(\sqrt{r}\varphi) \operatorname{sn}(\sqrt{r}\varphi_t) - k \operatorname{sn}(\sqrt{r}\varphi) \operatorname{dn}(\sqrt{r}\varphi_t),$$

$$\cos \frac{\theta_t}{2} = \operatorname{dn}(\sqrt{r}\varphi) \operatorname{dn}(\sqrt{r}\varphi_t) + k^2 \operatorname{sn}(\sqrt{r}\varphi) \operatorname{sn}(\sqrt{r}\varphi_t),$$

$$\begin{aligned} x_t = & \frac{2}{\sqrt{r}} \operatorname{dn}^2(\sqrt{r}\varphi) (E(\sqrt{r}\varphi_t) - E(\sqrt{r}\varphi)) + \\ & + \frac{4k^2}{\sqrt{r}} \operatorname{dn}(\sqrt{r}\varphi) \operatorname{sn}(\sqrt{r}\varphi) (\operatorname{cn}(\sqrt{r}\varphi) - \operatorname{cn}(\sqrt{r}\varphi_t)) + \\ & + \frac{2k^2}{\sqrt{r}} \operatorname{sn}^2(\sqrt{r}\varphi) (\sqrt{r}t + E(\sqrt{r}\varphi) - E(\sqrt{r}\varphi_t)) - t, \end{aligned}$$

$$\begin{aligned} y_t = & \frac{2k}{\sqrt{r}} (2 \operatorname{dn}^2(\sqrt{r}\varphi) - 1) (\operatorname{cn}(\sqrt{r}\varphi) - \operatorname{cn}(\sqrt{r}\varphi_t)) - \\ & - \frac{2k}{\sqrt{r}} \operatorname{sn}(\sqrt{r}\varphi) \operatorname{dn}(\sqrt{r}\varphi) (2E(\sqrt{r}\varphi_t) - 2E(\sqrt{r}\varphi) - \sqrt{r}t). \end{aligned}$$

# Extremal trajectories — Euler elasticae

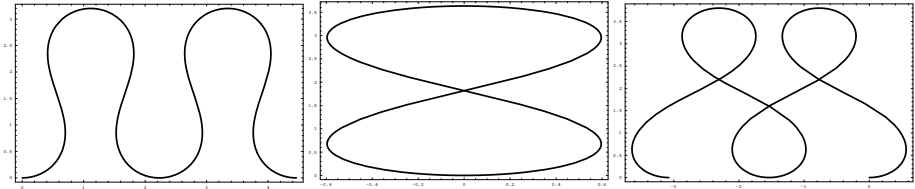


Figure: Inflectional

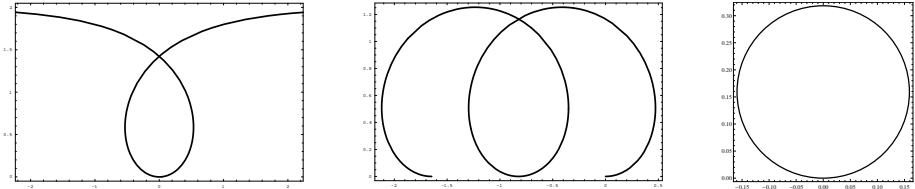
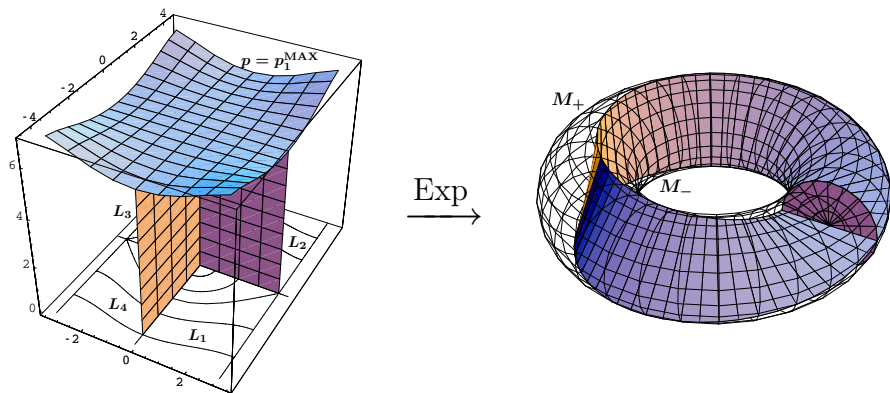


Figure: Critical

Figure: Non-inflectional

Figure: Degenerate

# Global structure of exponential mapping



Exp :  $L_1, L_3 \rightarrow M_+$  is diffeomorphism,

Exp :  $L_2, L_4 \rightarrow M_-$  is diffeomorphism.

# Visualization of elasticae (Epicycloid rotate)

Show movie

# Visualization of elasticae (Epicycloid wave)

Show movie

# Visualization of elasticae (Eight)

Show movie

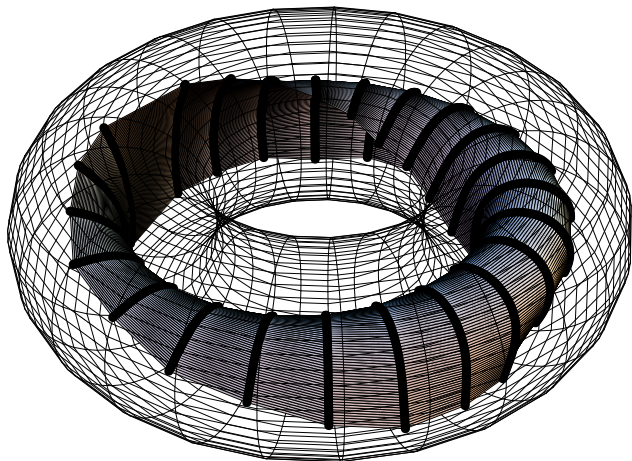
# Visualization of elasticae (Ellips wave)

Show movie

# Visualization of elasticae (Rotates)

Show movie

# Cut locus



# Visualization of globally optimal elastica

Show movie

# Visualization of locally optimal elasticae 1

Show movie

# Visualization of locally optimal elasticae 2

Show movie

# Results on Part 1

- 1 New parametrization of extremal trajectories was obtained.
- 2 The software was developed
  - Parallel and sequential computing of globally and locally optimal curves.
  - Plots of extremal and optimal elasticae.
  - Construction of cut locus.
  - Parallel and sequential computing of animations of globally and locally optimal elasticae.

## Part 2: Statement of sub-Riemannian problem on the group of motions of a plane

$$\dot{q} = \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \\ \dot{v} \end{pmatrix} = u_1 \begin{pmatrix} 1 \\ 0 \\ -\frac{y}{2} \\ 0 \end{pmatrix} + u_2 \begin{pmatrix} 0 \\ 1 \\ \frac{x}{2} \\ \frac{x^2+y^2}{2} \end{pmatrix},$$

$$q = (x, y, z, v) \in \mathbb{R}^4, \quad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$q(0) = q_0 = (0, 0, 0, 0)^T, \quad q(t_1) = q_1 = (x_1, y_1, z_1, v_1)^T,$$

$$\int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min \iff \int_0^{t_1} \frac{u_1^2 + u_2^2}{2} dt \rightarrow \min.$$

# Geometric formulation of the problem

**Given:**

$$a_0, a_1 \in \mathbb{R}^2,$$

$\gamma_0 \subset \mathbb{R}^2$  connecting  $a_1$  to  $a_0$

$$S \in \mathbb{R}, \quad \text{line } L \subset \mathbb{R}^2.$$

**Find:**

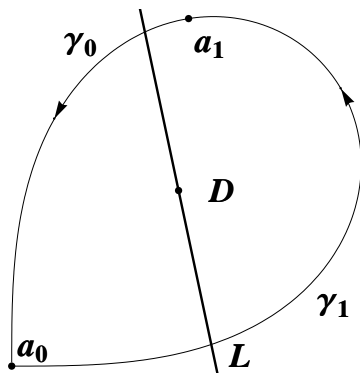
$\gamma_1 \subset \mathbb{R}^2$  connecting  $a_0$  to  $a_1$ ,

$$\text{s. t. } \gamma_1 \cup \gamma_0 = \partial D,$$

$$\text{area}(D) = S,$$

center of mass of  $D \in L$ ,

$$\text{length}(\gamma_1) \rightarrow \min.$$



# Overview

- Parameterization of extremal curves.
- Symmetries of exponential mapping and construction of the Maxwell sets.
- Global bound of the cut time and necessary optimality conditions for extremal curves.
- Algorithm and software for numerical solution of the problem.

# Known results for invariant sub-Riemannian problems on Lie groups

- 1 Three-dimensional Lie groups:
  - Heisenberg group (A. M. Vershik, V. Ya. Gershkovich 1986),
  - $SL(2)$ ,  $SO(3)$ ,  $S^3$  (U. Boscain, F. Rossi 2008),
  - $SE(2)$  (Yu. L. Sachkov 2010)
  
- 2 5-dimensional nilpotent Lie group with growth vector  $(2, 3, 5)$  (Yu.L.Sachkov 2006).
  
- 3 6-dimensional nilpotent Lie group with growth vector  $(3, 6)$  (O.M. Myasnichenko 2002).

# Nilpotent sub-Riemannian problem on the Engel group

$$X_1 = (1, 0, -\frac{y}{2}, 0)^T, \quad X_2 = (0, 1, \frac{x}{2}, \frac{x^2 + y^2}{2})^T.$$

$$\text{Lie}(X_1, X_2) = \text{span}(X_1, X_2, X_3, X_4),$$

$$\dim \text{Lie}(X_1, X_2)(q) = 4,$$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4,$$

$$[X_1, X_4] = [X_2, X_3] = [X_2, X_4] = 0.$$

Growth vector (2, 3, 4).

Nilpotent approximation of nonholonomic control systems in four-dimensional space with two-dimensional control (e. g. car with trailer).

# Controllability and existence of optimal curves

- 1  $X_1(q), \dots, X_4(q)$  are linearly independent  
 $\forall q \in \mathbb{R}^4 \xrightarrow{\text{Rashevskii-Chow theorem}}$  complete controllability.
- 2 Existence of optimal solutions is implied by Filippov theorem.

# Pontryagin's maximum principle :

## Abnormal extremal trajectories

$$x = 0, \quad y = \pm t, \quad z = 0, \quad v = \pm \frac{t^3}{6}.$$

## Normal Hamiltonian system

$$\begin{aligned}\dot{\theta} &= c, & \theta &\in S^1, \\ \dot{c} &= -\alpha \sin \theta, & c &\in \mathbb{R}, \\ \dot{\alpha} &= 0, & \alpha &\in \mathbb{R}, \\ \dot{x} &= -\sin \theta, \\ \dot{y} &= \cos \theta, \\ \dot{z} &= \frac{x \cos \theta + y \sin \theta}{2}, \\ \dot{v} &= \cos \theta \frac{x^2 + y^2}{2}.\end{aligned}$$

$$E = \frac{c^2}{2} - \alpha \cos \theta \in [-|\alpha|, +\infty).$$

## Equation of pendulum and physical meaning of $\alpha$

$$\ddot{\theta} = -\alpha \sin \theta, \quad \alpha = \frac{g}{L} = \text{const} \in \mathbb{R}$$

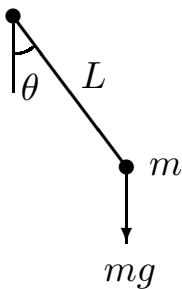


Figure: Mathematical pendulum with  $\alpha > 0$

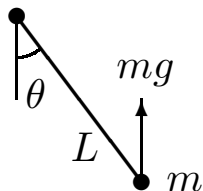


Figure: Mathematical pendulum with  $\alpha < 0$

# Stratification of phase cylinder of pendulum

$$C = T_{q_0}^* M \cap \{H = 1/2\} = \{\lambda = (\theta, c, \alpha) \mid \theta \in S^1, c, \alpha \in \mathbb{R}\}.$$

$$C = \cup_{i=1}^7 C_i, \quad C_i \cap C_j = \emptyset, \quad i \neq j.$$

$$C_i^+ = C_i \cap \{\alpha > 0\}, \quad C_i^- = C_i \cap \{\alpha < 0\}, \quad i \in \{1, \dots, 5\},$$

$$C_{i+}^\pm = C_i^\pm \cap \{c > 0\}, \quad C_{i-}^\pm = C_i^\pm \cap \{c < 0\}, \quad i \in \{2, 3\}.$$

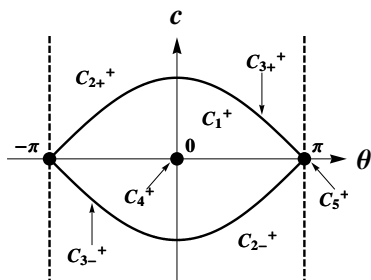


Figure: Stratification for  $\alpha > 0$

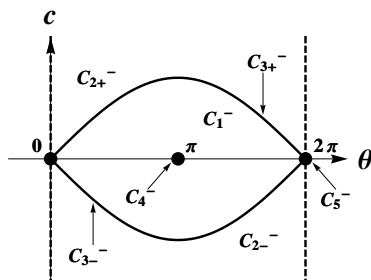


Figure: Stratification for  $\alpha < 0$

## Elliptic coordinates in $C^+$

$$\lambda \in C_1^+,$$

$$k = \sqrt{\frac{E + \alpha}{2\alpha}} = \sqrt{\frac{c^2}{4\alpha} + \sin^2 \frac{\theta}{2}} \in (0, 1),$$

$$\sin \frac{\theta}{2} = k \operatorname{sn}(\sqrt{\alpha}\varphi),$$

$$\frac{c}{2} = k\sqrt{\alpha} \operatorname{cn}(\sqrt{\alpha}\varphi),$$

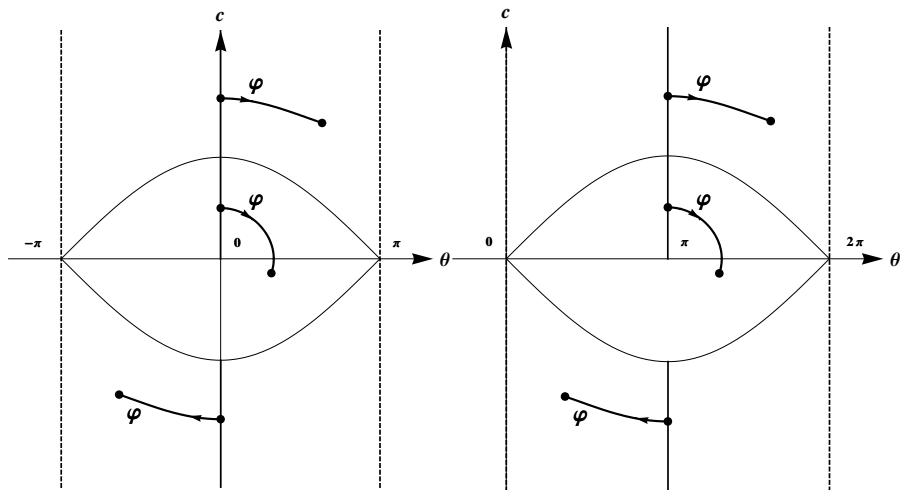
$$\cos \frac{\theta}{2} = \operatorname{dn}(\sqrt{\alpha}\varphi),$$

$$\varphi \in [0, 4K],$$

where  $\operatorname{sn}$ ,  $\operatorname{cn}$ ,  $\operatorname{dn}$ ,  $E$  are elliptic Jacobi's functions.

$$\text{Equation of pendulum: } \dot{\varphi} = 1, \quad \dot{k} = \dot{\alpha} = 0.$$

# Elliptic coordinates $(\varphi, k)$ in the phase cylinder of pendulum



# Elliptic coordinates in $C^-$

Coordinates in the sets  $C_1^-$ ,  $C_2^-$ ,  $C_3^-$ :

$$\varphi(\theta, c, \alpha) = \varphi(\theta - \pi, c, -\alpha),$$

$$k(\theta, c, \alpha) = k(\theta - \pi, c, -\alpha).$$

# Parametrization of extremal curves in the case $\alpha = 1$

$\lambda \in C_1^+$  (oscillations of pendulum)  $\Rightarrow$

$$x_t = 2k(\operatorname{cn} \varphi_t - \operatorname{cn} \varphi),$$

$$y_t = 2(E(\varphi_t) - E(\varphi)) - t,$$

$$z_t = 2k(\operatorname{sn} \varphi_t \operatorname{dn} \varphi_t - \operatorname{sn} \varphi \operatorname{dn} \varphi - \frac{y_t}{2}(\operatorname{cn} \varphi_t + \operatorname{cn} \varphi)),$$

$$\begin{aligned} v_t = & \frac{y_t^3}{6} + 2k^2 \operatorname{cn}^2 \varphi y_t - 4k^2 \operatorname{cn} \varphi (\operatorname{sn} \varphi_t \operatorname{dn} \varphi_t - \operatorname{sn} \varphi \operatorname{dn} \varphi) + \\ & + 2k^2 \left( \frac{2}{3} \operatorname{cn} \varphi_t \operatorname{dn} \varphi_t \operatorname{sn} \varphi_t - \frac{2}{3} \operatorname{cn} \varphi \operatorname{dn} \varphi \operatorname{sn} \varphi + \frac{1 - k^2}{3k^2} t + \right. \\ & \left. + \frac{2k^2 - 1}{3k^2} (E(\varphi_t) - E(\varphi)) \right). \end{aligned}$$

# Symmetries of Hamiltonian system

Dilation of  $\alpha$ :

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto \left(\theta, \frac{c}{\sqrt{\alpha}}, 1, \sqrt{\alpha}x, \sqrt{\alpha}y, \alpha z, \alpha^{\frac{3}{2}}v, \sqrt{\alpha}t\right),$$

$$(\varphi, k, \alpha) \mapsto (\sqrt{\alpha}\varphi, k, 1).$$

Inversion of  $\alpha$ :

$$(\theta, c, \alpha, x, y, z, v, t) \mapsto (\theta - \pi, c, -\alpha, -x, -y, z, -v, t),$$

$$(\varphi, k, \alpha) \mapsto (\varphi, k, -\alpha).$$

# Parametrization of extremal trajectories in general case with $\lambda \in \cup_{i=1}^3 C_i$

$$(x_t, y_t, z_t, v_t)(\varphi, k, \alpha) = \left( \frac{s_1}{\sigma} x_{\sigma t}, \frac{s_1}{\sigma} y_{\sigma t}, \frac{1}{\sigma^2} z_{\sigma t}, \frac{s_1}{\sigma^3} v_{\sigma t} \right) (\sigma \varphi, k, 1),$$

where  $\sigma = \sqrt{|\alpha|}$ ,  $s_1 = \operatorname{sgn} \alpha$ .

## General case with $\alpha \neq 0$

$$\lambda \in C_1 \Rightarrow$$

$$x_t = \frac{2k\sigma}{\alpha} (\operatorname{cn}(\sigma\varphi_t) - \operatorname{cn}(\sigma\varphi)),$$

$$y_t = \frac{2\sigma}{\alpha} (E(\sigma\varphi_t) - E(\sigma\varphi)) - \operatorname{sgn} \alpha t,$$

$$z_t = \frac{2k}{|\alpha|} (\operatorname{sn}(\sigma\varphi_t) \operatorname{dn}(\sigma\varphi_t) - \operatorname{sn}(\sigma\varphi) \operatorname{dn}(\sigma\varphi) - \\ - \frac{\sigma k y_t}{2\alpha} (\operatorname{cn}(\sigma\varphi_t) + \operatorname{cn}(\sigma\varphi))),$$

$$v_t = \dots$$

## Parametrization of extremal curves for degenerate cases

$$\lambda \in C_4 \Rightarrow x_t = 0, \quad y_t = t \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = \frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_5 \Rightarrow x_t = 0, \quad y_t = -t \operatorname{sgn} \alpha, \quad z_t = 0, \quad v_t = -\frac{t^3}{6} \operatorname{sgn} \alpha.$$

$$\lambda \in C_6 \Rightarrow$$

$$x_t = \frac{\cos(ct + \theta) - \cos \theta}{c}, \quad y_t = \frac{\sin(ct + \theta) - \sin \theta}{c},$$
$$z_t = \frac{ct - \sin(ct)}{2c^2}, \quad v_t = -\frac{2c \cos \theta t - 4 \sin(ct + \theta) + \sin(2ct + \theta)}{4c^3}.$$

$$\lambda \in C_7 \Rightarrow x_t = -t \sin \theta, \quad y_t = t \cos \theta, \quad z_t = 0, \quad v_t = \frac{\cos \theta}{6} t^3.$$

## Exponential mapping, Maxwell points and cut time

$$\text{Exp} : \mathcal{C} \times \mathbb{R}_+ \rightarrow M = \mathbb{R}^4,$$

$$\text{Exp}(\lambda, t) = q_t,$$

$$\lambda = (\theta, c, \alpha) \in \mathcal{C}, \quad t \in \mathbb{R}_+, \quad q_t \in M.$$

$$\text{MAX} = \{(\lambda, t) \mid \exists \tilde{\lambda} \neq \lambda, \text{Exp}(\lambda, t) = \text{Exp}(\tilde{\lambda}, t)\},$$

$$t_{\text{cut}}(\lambda) = \sup\{t > 0 \mid \text{Exp}(\lambda, s) \text{ is optimal for } s \in [0, t]\},$$

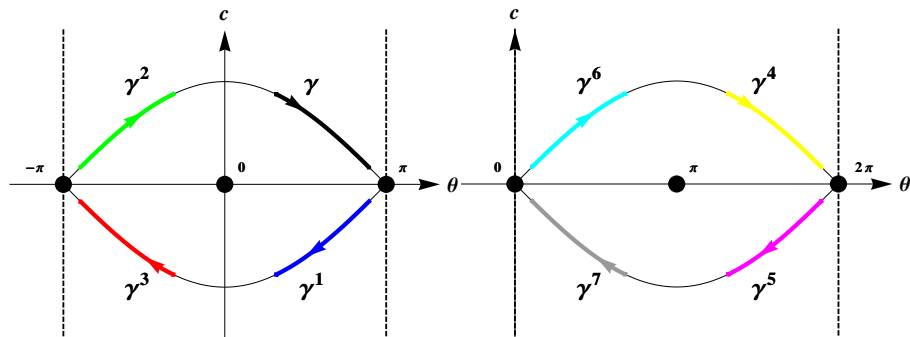
$$t_{\text{cut}}(\lambda) \leq t \text{ for any } (\lambda, t) \in \text{MAX}.$$

# Group of symmetries of exponential mapping

$G$	$\varepsilon^1$	$\varepsilon^2$	$\varepsilon^3$	$\varepsilon^4$	$\varepsilon^5$	$\varepsilon^6$	$\varepsilon^7$
$\varepsilon^1$	ld	$\varepsilon^3$	$\varepsilon^2$	$\varepsilon^5$	$\varepsilon^4$	$\varepsilon^7$	$\varepsilon^6$
$\varepsilon^2$		ld	$\varepsilon^1$	$\varepsilon^6$	$\varepsilon^7$	$\varepsilon^4$	$\varepsilon^5$
$\varepsilon^3$			ld	$\varepsilon^7$	$\varepsilon^6$	$\varepsilon^5$	$\varepsilon^4$
$\varepsilon^4$				ld	$\varepsilon^1$	$\varepsilon^2$	$\varepsilon^3$
$\varepsilon^5$					ld	$\varepsilon^3$	$\varepsilon^2$
$\varepsilon^6$						ld	$\varepsilon^1$
$\varepsilon^7$							ld

Table: Multiplication in  $G = \{\text{ld}, \varepsilon^1, \varepsilon^2, \varepsilon^3, \varepsilon^4, \varepsilon^5, \varepsilon^6, \varepsilon^7\}$

# Reflections of trajectories of pendulum



## Reflections of trajectories of pendulum

$$\varepsilon^1 : \gamma \mapsto \gamma^1 = \{(\theta_s^1, c_s^1, \alpha^1)\} = \{(\theta_{t-s}, -c_{t-s}, \alpha) \mid s \in [0, t]\},$$

$$\varepsilon^2 : \gamma \mapsto \gamma^2 = \{(\theta_s^2, c_s^2, \alpha^2)\} = \{(-\theta_{t-s}, c_{t-s}, \alpha) \mid s \in [0, t]\},$$

$$\varepsilon^3 : \gamma \mapsto \gamma^3 = \{(\theta_s^3, c_s^3, \alpha^3)\} = \{(-\theta_s, -c_s, \alpha) \mid s \in [0, t]\},$$

$$\varepsilon^4 : \gamma \mapsto \gamma^4 = \{(\theta_s^4, c_s^4, \alpha^4)\} = \{(\theta_s + \pi, c_s, -\alpha) \mid s \in [0, t]\},$$

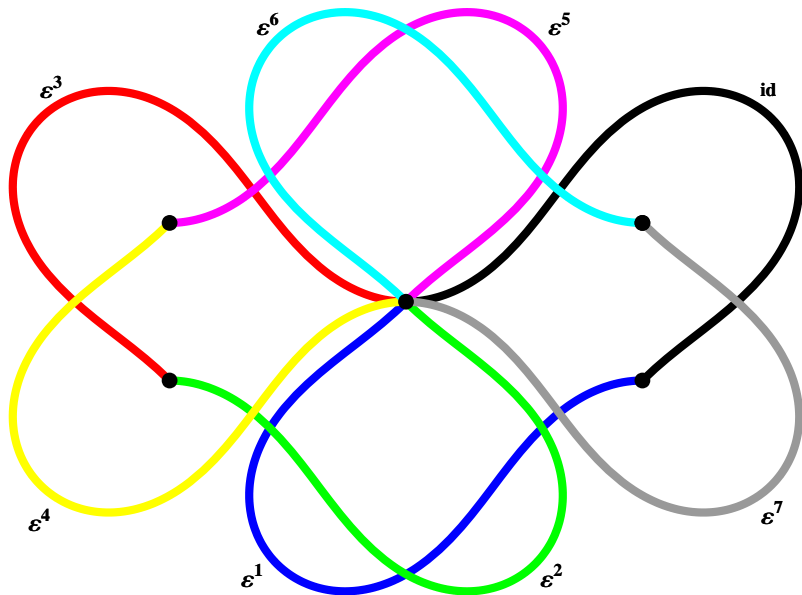
$$\varepsilon^5 : \gamma \mapsto \gamma^5 = \{(\theta_s^5, c_s^5, \alpha^5)\} = \{(\theta_{t-s} + \pi, -c_{t-s}, -\alpha) \mid s \in [0, t]\},$$

$$\varepsilon^6 : \gamma \mapsto \gamma^6 = \{(\theta_s^6, c_s^6, \alpha^6)\} = \{(-\theta_{t-s} + \pi, c_{t-s}, -\alpha) \mid s \in [0, t]\},$$

$$\varepsilon^7 : \gamma \mapsto \gamma^7 = \{(\theta_s^7, c_s^7, \alpha^7)\} = \{(-\theta_s + \pi, -c_s, -\alpha) \mid s \in [0, t]\},$$

where  $\gamma = \{(\theta_s, c_s, \alpha) \mid s \in [0, t]\}$ .

# Reflections of Euler elasticae



## Action of $\varepsilon^i$ in the preimage of exponential mapping

$$\varepsilon^i : \mathbb{C} \times \mathbb{R} \rightarrow \mathbb{C} \times \mathbb{R},$$

$$(\theta^1, c^1, \alpha^1) = (\theta_t, -c_t, \alpha),$$

$$(\theta^3, c^3, \alpha^3) = (-\theta_t, -c_t, \alpha),$$

$$(\theta^5, c^5, \alpha^5) = (\theta_t + \pi, -c_t, -\alpha),$$

$$(\theta^7, c^7, \alpha^7) = (-\theta_t + \pi, -c_t, -\alpha).$$

$$\varepsilon^i(\theta, c, \alpha, t) = (\theta^i, c^i, \alpha^i, t),$$

$$(\theta^2, c^2, \alpha^2) = (-\theta_t, c_t, \alpha),$$

$$(\theta^4, c^4, \alpha^4) = (\theta_t + \pi, c_t, -\alpha),$$

$$(\theta^6, c^6, \alpha^6) = (-\theta_t + \pi, c_t, -\alpha),$$

## Action of $\varepsilon^i$ in the image of exponential mapping

$$\varepsilon^i : M \rightarrow M, \quad \varepsilon^i(q) = \varepsilon^i(x, y, z, v) = q^i = (x^i, y^i, z^i, v^i),$$

$$(x^1, y^1, z^1, v^1) = (x, y, -z, v - xz),$$

$$(x^2, y^2, z^2, v^2) = (-x, y, z, v - xz),$$

$$(x^3, y^3, z^3, v^3) = (-x, y, -z, v),$$

$$(x^4, y^4, z^4, v^4) = (-x, y, -z, -v),$$

$$(x^5, y^5, z^5, v^5) = (-x, -y, -z, -v + xz),$$

$$(x^6, y^6, z^6, v^6) = (x, -y, z, -v + xz),$$

$$(x^7, y^7, z^7, v^7) = (x, -y, -z, -v).$$

# Reflections as symmetries of $\text{Exp}$

## Proposition

Reflection  $\varepsilon^i$  is a symmetry of exponential mapping for any  $i = 1, \dots, 7$ , i. e.,

$$\varepsilon^i \circ \text{Exp}(\theta, c, \alpha, t) = \text{Exp} \circ \varepsilon^i(\theta, c, \alpha, t),$$
$$(\theta, c, \alpha) \in C, \quad t \in \mathbb{R}_+.$$

$$\text{MAX}^i = \{(\lambda, t) \in C \times \mathbb{R}_+ \mid \lambda^i \neq \lambda, \text{Exp}(\lambda^i, t) = \text{Exp}(\lambda, t)\},$$
$$\lambda = (\theta, c, \alpha), \quad \lambda^i = (\theta^i, c^i, \alpha^i) = \varepsilon^i(\lambda).$$

# Fixed points of $\varepsilon^i$ in the image of exponential mapping

$$\text{Exp}(\lambda^i, t) = \text{Exp}(\lambda, t) \iff \varepsilon^i(q_t) = q_t.$$

## Lemma

- 1  $\varepsilon^1(q) = q \iff z = 0,$
- 2  $\varepsilon^2(q) = q \iff x = 0,$
- 3  $\varepsilon^3(q) = q \iff x^2 + z^2 = 0,$
- 4  $\varepsilon^4(q) = q \iff x^2 + y^2 + v^2 = 0,$
- 5  $\varepsilon^5(q) = q \iff x^2 + y^2 + z^2 + v^2 = 0,$
- 6  $\varepsilon^6(q) = q \iff y^2 + (2v - xz)^2 = 0,$
- 7  $\varepsilon^7(q) = q \iff y^2 + z^2 + v^2 = 0.$

# Fixed points of $\varepsilon^i$ in the preimage of exponential mapping

## Proposition

If  $(\lambda, t) \in C \times \mathbb{R}_+$ ,  $\varepsilon^i(\lambda, t) = (\lambda^i, t)$  then:

$$\textcircled{1} \lambda^1 = \lambda \iff \begin{cases} \text{cn } \tau = 0 \text{ if } \lambda \in C_1 \\ \text{is impossible if } \lambda \in C_2 \cup C_3 \cup C_6 \end{cases}$$

$$\textcircled{2} \lambda^2 = \lambda \iff \begin{cases} \text{sn } \tau = 0 \text{ if } \lambda \in C_1 \\ \text{sn } \tau \text{ cn } \tau = 0 \text{ if } \lambda \in C_2 \\ \tau = 0 \text{ if } \lambda \in C_3 \\ 2\theta + ct = 2\pi n \text{ if } \lambda \in C_6 \end{cases}$$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad \tau = \sigma \frac{\varphi + \varphi t}{2k}.$$

# Complete description of the Maxwell sets for $\varepsilon^1, \varepsilon^2$

## Theorem

- 1  $\text{MAX}^1 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = p_z^n(k), n \in \mathbb{N}, \text{cn}(\tau) \neq 0\}$ ,
- 2  $\text{MAX}^1 \cap N_2 = \text{MAX}^1 \cap N_3 = \text{MAX}^1 \cap N_6 = \emptyset$ ,
- 3  $\text{MAX}^2 \cap N_1 = \{(\lambda, t) \in N_1 \mid p = 2Kn, n \in \mathbb{N}, \text{sn}(\tau) \neq 0\}$ ,
- 4  $\text{MAX}^2 \cap N_2 = \{(\lambda, t) \in N_2 \mid p = Kn, n \in \mathbb{N}, \text{sn}(\tau) \text{cn}(\tau) \neq 0\}$ ,
- 5  $\text{MAX}^2 \cap N_3 = \emptyset$ ,
- 6  $\text{MAX}^2 \cap N_6 = \{(\lambda, t) \in N_6 \mid tc = 2\pi n, \theta \neq \pi k, n, k \in \mathbb{Z}\}$

$$(\lambda, t) \in C_1 \cup C_3 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2},$$

$$(\lambda, t) \in C_2 \times \mathbb{R}_+ \quad \Rightarrow \quad p = \frac{\sigma t}{2k}.$$

$p_z^n(k) > 0$  —  $n$ -th root of  $\text{dn}(p) \text{sn}(p) + (p - 2E(p)) \text{cn}(p) = 0$ .

## Bound of the cut time

$$\lambda \in C_1 \Rightarrow \mathbf{t} = \min(2p_z^1, 4K)\sigma,$$

$$\lambda \in C_2 \Rightarrow \mathbf{t} = 2Kk\sigma,$$

$$\lambda \in C_6 \Rightarrow \mathbf{t} = \frac{2\pi}{|c|},$$

$$\lambda \in C_3 \cup C_4 \cup C_5 \cup C_7 \Rightarrow \mathbf{t} = +\infty.$$

Theorem (A. A., Yu. Sachkov)

For any  $\lambda \in C$

$$t_{cut}(\lambda) \leq \mathbf{t}(\lambda)$$

# Numerical solution of the optimal control problem: Reduction to the system of equations

$Y = \frac{y_t}{x_t}$ ,  $Z = \frac{z_t}{x_t^2}$ ,  $V = \frac{v_t}{x_t^3}$  are independent on  $\alpha$ .

$$Y_1 = \frac{y_1}{x_1}, \quad z_1 = \frac{z_1}{x_1^2}, \quad V_1 = \frac{v_1}{x_1^3}.$$

$$\begin{cases} Y(\tau, p, k) = Y_1, \\ Z(\tau, p, k) = Z_1, \\ V(\tau, p, k) = V_1. \end{cases}$$

## Decomposition of the preimage of exponential mapping

$$C = \cup_{i=1}^4 D_i,$$

$$D_1 \cap C_1 = \{\tau \in (0, K), p \in (0, p_{min}^1), k \in (0, 1)\},$$

$$D_1 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\},$$

$$D_2 \cap C_1 = \{\tau \in (K, 2K), p \in (0, p_{min}^1), k \in (0, 1)\},$$

$$D_2 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\},$$

$$D_3 \cap C_1 = \{\tau \in (2K, 3K), p \in (0, p_{min}^1), k \in (0, 1)\},$$

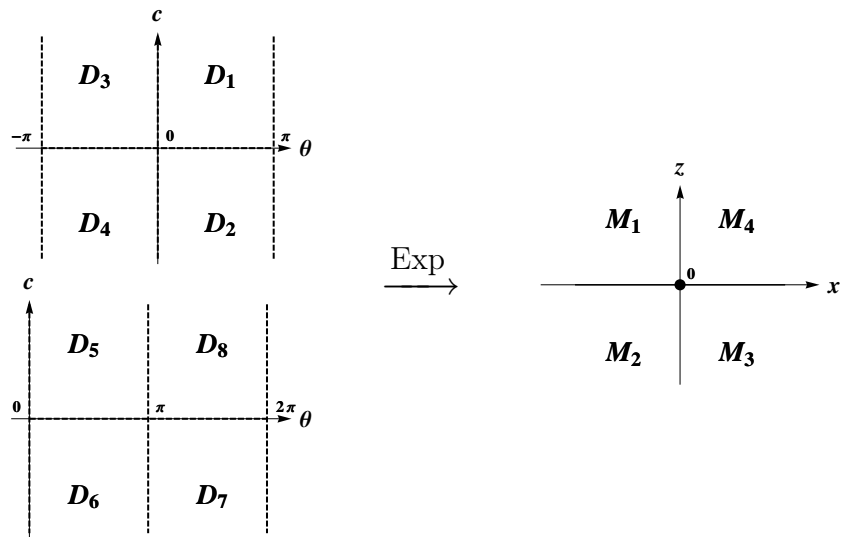
$$D_3 \cap C_2 = \{\tau \in (0, K), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\},$$

$$D_4 \cap C_1 = \{\tau \in (3K, 4K), p \in (0, p_{min}^1), k \in (0, 1)\},$$

$$D_4 \cap C_2 = \{\tau \in (-K, 0), p \in (0, K), k \in (0, 1), \operatorname{sgn} c = 1\},$$

where  $p_{min}^1 = \min(p_z^1, 2K)$ .

# Correspondence between domains in image and preimage of exponential mapping



Conjecture:  $\text{Exp} : D_i \rightarrow M_i$  and  $\text{Exp} : D_{i+4} \rightarrow M_i$  are diffeomorphisms.

## Results on Part 2

- Nilpotent sub-Riemannian problem on the Engel group was considered.
- Extremal curves for this problem were found.
- Symmetries of exponential mapping and the corresponding Maxwell points were computed.
- Global upper bound of the cut time along extremal curves was proved.
- Problem was reduced to solving of the system of three algebraic equations.
- Development of the software for numerical solution of the problem was started.

# Plans

- Complete investigation of optimality of extremal curves and developing of the program for computing optimal curves for sub-Riemannian problem on the Engel group.
- Nilpotent approximation of nonholonomic systems in four-dimensional space with two-dimensional control (in particular, car with the trailer).