

Exponential mapping in Euler's elastic problem

Yuri Sachkov

Program Systems Institute
Russian Academy of Sciences
Pereslavl-Zalessky, Russia
E-mail: sachkov@sys.botik.ru

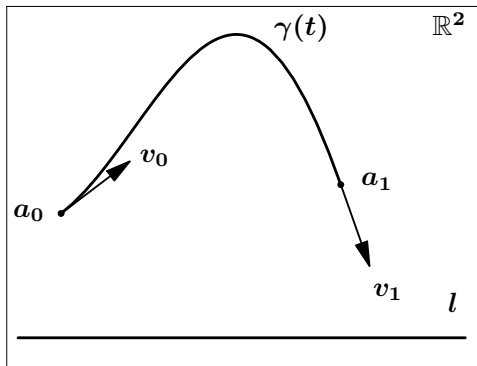
Mathematical Control Theory and
Mechanics

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Suzdal, June 22 – June 27, 2007

Summary of the talk

- 1 Statement and history of the problem
- 2 Optimal control problem
- 3 Attainable set and existence of optimal solutions
- 4 Extremal trajectories
- 5 Local and global optimality of extremal trajectories
- 6 Stability of elasticae
- 7 Global structure of exponential mapping
- 8 Movies

Problem statement: Stationary configurations of elastic rod



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.

Find: $\gamma(t)$, $t \in [0, t_1]$:

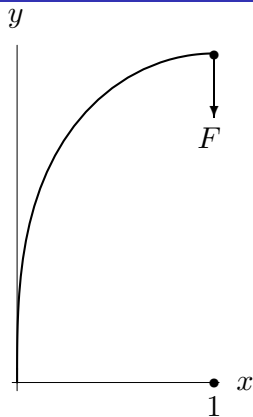
$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot{\gamma}(0) = v_0$, $\dot{\gamma}(t_1) = v_1$. $|\dot{\gamma}(t)| \equiv 1 \Rightarrow t_1 = l$

Elastic energy $J = \frac{1}{2} \int_0^{t_1} k^2 dt \rightarrow \min$,

$k(t)$ — curvature of $\gamma(t)$.

1691: James (Jacob) Bernoulli

Rectangular elastica:



$$dy = \frac{x^2 dx}{\sqrt{1-x^4}}, \quad ds = \frac{dx}{\sqrt{1-x^4}}, \quad x \in [0, 1]$$

Integration in series

- Elastic energy

$$E = \text{const} \cdot \int \frac{ds}{R^2},$$

R — radius of curvature,

- Letter to Leonhard Euler: proposal of the variational problem

$$E \rightarrow \min .$$

- “Methodus inveniendi lineas curvas maximi minimive proprietate gaudentes, sive Solutio problematis isoperimetrici latissimo sensu accepti”, Lausanne, Geneva, 1744,
- Appendix “De curvis elasticis”,
- *“That among all curves of the same length which not only pass through the points A and B, but are also tangent to given straight lines at these points, that curve be determined in which the value of $\int_A^B \frac{ds}{R^2}$ be a minimum.”*

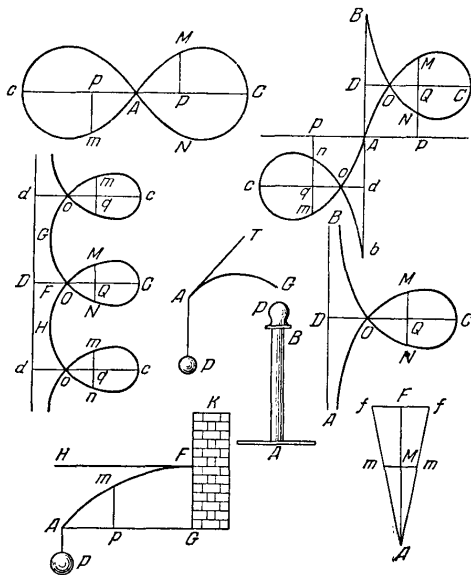
- Problem of calculus of variations,
- Euler-Lagrange equation,
- Reduction to quadratures

$$dy = \frac{(\alpha + \beta x + \gamma x^2) dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

$$ds = \frac{a^2 dx}{\sqrt{a^4 - (\alpha + \beta x + \gamma x^2)^2}},$$

- Qualitative analysis of the integrals
- Types of solutions (elasticae)

Euler's sketches



Explicit parametrization of Euler elasticae by Jacobi's functions

- Ph.D. thesis “Stability of elastic lines in the plane and the space”
- Euler-Lagrange equation \Rightarrow

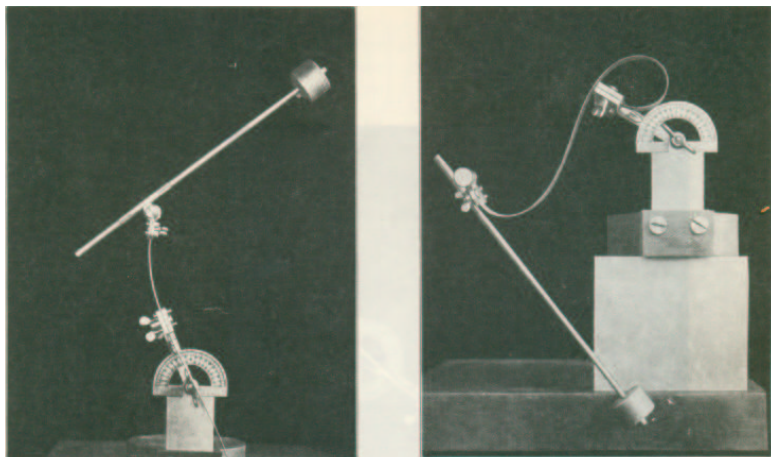
$$\begin{aligned}\dot{x} &= \cos \theta, & \dot{y} &= \sin \theta, \\ A\ddot{\theta} + B \sin(\theta - \gamma) &= 0, & A, B, \gamma &= \text{const},\end{aligned}$$

equation of pendulum,

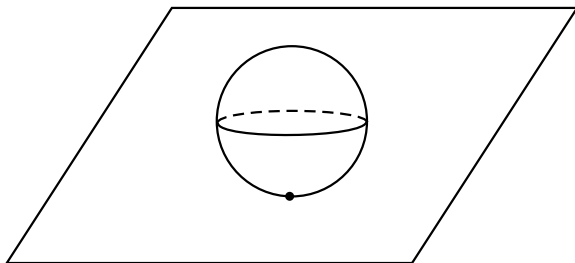
- elastic arc without inflection points \Rightarrow stable,
- elastic arc with inflection points \Rightarrow numerical investigation,
- numeric plots of elasticae.

1906: Max Born

Experiments on elastic rods:



Euler elasticae in the ball-plate problem



Euler elasticae in the nilpotent sub-Riemannian problem with the growth vector (2,3,5):

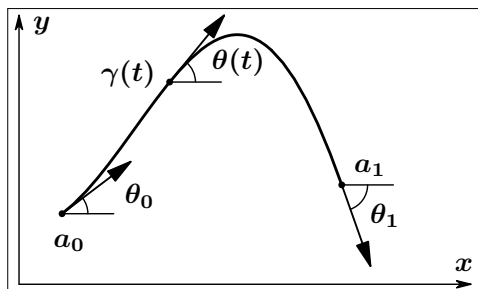
$$\dot{q} = u_1 X_1 + u_2 X_2, \quad q \in \mathbb{R}^5, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0, \quad q(t_1) = q_1,$$

$$I = \int_0^{t_1} \sqrt{u_1^2 + u_2^2} dt \rightarrow \min,$$

$$[X_1, X_2] = X_3, \quad [X_1, X_3] = X_4, \quad [X_2, X_3] = X_5.$$

Euler's problem: Coordinates in $\mathbb{R}^2 \times S^1$



- $(x, y) \in \mathbb{R}^2$, $\theta \in S^1$,
- $\gamma(t) = (x(t), y(t))$, $t \in [0, t_1]$,
- $a_0 = (x_0, y_0)$, $a_1 = (x_1, y_1)$,
- $v_0 = (\cos \theta_0, \sin \theta_0)$, $v_1 = (\cos \theta_1, \sin \theta_1)$.

Optimal control problem

$$\dot{x} = \cos \theta,$$

$$\dot{y} = \sin \theta,$$

$$\dot{\theta} = u,$$

$$q = (x, y, \theta) \in \mathbb{R}_{x,y}^2 \times \mathcal{S}_\theta^1, \quad u \in \mathbb{R},$$

$$q(0) = q_0 = (x_0, y_0, \theta_0), \quad q(t_1) = q_1 = (x_1, y_1, \theta_1), \quad t_1 \text{ fixed.}$$

$$k^2 = \dot{\theta}^2 = u^2 \quad \Rightarrow \quad J = \frac{1}{2} \int_0^{t_1} u^2 dt \rightarrow \min.$$

Admissible controls $u(t) \in L_2[0, t_1]$,
trajectories $q(t) \in AC[0, t_1]$

Left-invariant problem on the group of motions of a plane

$$E(2) = \mathbb{R}^2 \rtimes SO(2) = \left\{ \begin{pmatrix} \cos \theta & -\sin \theta & x \\ \sin \theta & \cos \theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid (x, y) \in \mathbb{R}^2, \theta \in S^1 \right\}$$

$$\dot{q} = X_1(q) + uX_2(q), \quad q \in E(2), \quad u \in \mathbb{R}.$$

$$q(0) = q_0, \quad q(t_1) = q_1, \quad t_1 \text{ fixed},$$

$$J = \frac{1}{2} \int_0^{t_1} u^2 dt \rightarrow \min,$$

Left-invariant frame on $E(2)$:

$$X_1(q) = qE_{13}, \quad X_2(q) = q(E_{21} - E_{12}), \quad X_3(q) = -qE_{23}$$

Continuous symmetries and normalization of conditions of the problem

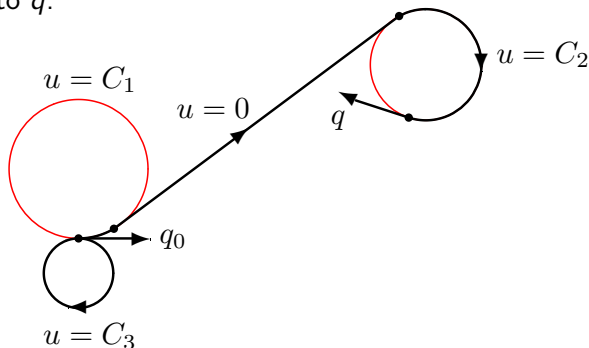
- Left translations on $E(2) \Rightarrow q_0 = \text{Id} \in E(2)$:
 - Parallel translations in $\mathbb{R}^2 \Rightarrow (x_0, y_0) = (0, 0)$
 - Rotations in $\mathbb{R}^2 \Rightarrow \theta_0 = 0$
- Dilations in $\mathbb{R}^2 \Rightarrow t_1 = 1$

Attainable set

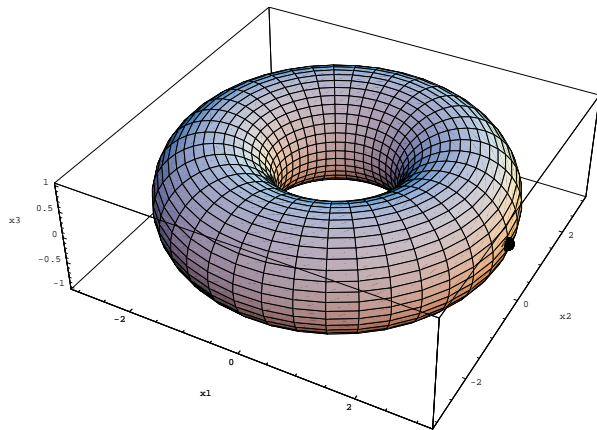
$$q_0 = \text{Id} = (0, 0, 0), \quad t_1 = 1$$

$$\mathcal{A}_{q_0}(1) = \{(x, y, \theta) \mid x^2 + y^2 < 1 \ \forall \theta \in S^1 \text{ or } (x, y, \theta) = (1, 0, 0)\}.$$

Steering q_0 to q :



Attainable set



In the sequel: $q_1 \in \mathcal{A}_{q_0}(t_1)$

Existence and regularity of optimal solutions

$$\begin{aligned}\dot{q} &= X_1(q) + uX_2(q), & q &\in \mathbb{R}^2 \times S^1, & u &\in \mathbb{R} \text{ unbounded} \\ q(0) &= q_0, & q(t_1) &= q_1, \\ J &= \frac{1}{2} \int_0^{t_1} u^2 dt \rightarrow \min,\end{aligned}$$

- General existence theorem $\Rightarrow \exists$ optimal $u(t) \in L_2$
- Compactification of the space of control parameters
 $\Rightarrow \exists$ optimal $u(t) \in L_\infty$

\Rightarrow Pontryagin Maximum Principle applicable

Pontryagin Maximum Principle in invariant form

$$\dot{q} = X_1(q) + uX_2(q), \quad q \in M = \mathbb{R}^2 \times S^1, \quad u \in \mathbb{R}, \quad J = \frac{1}{2} \int_0^{t_1} u^2 dt \rightarrow \min$$

- $T_q M = \text{span}(X_1(q), X_2(q), X_3(q)), \quad X_3 = [X_1, X_2]$
- $T_q^* M = \{(h_1, h_2, h_3)\}, \quad h_i(\lambda) = \langle \lambda, X_i \rangle, \quad \lambda \in T^* M$
- Hamiltonian vector fields $\vec{h}_i \in \text{Vec}(T^* M)$
- $h_u^\nu = \langle \lambda, X_1 + uX_2 \rangle + \frac{\nu}{2} u^2 = h_1(\lambda) + u h_2(\lambda) + \frac{\nu}{2} u^2$

Theorem (Pontryagin Maximum Principle)

$u(t)$ and $q(t)$ optimal $\Rightarrow \exists \lambda_t \in T_{q(t)}^* M, \nu \leq 0$:

$$\dot{\lambda}_t = \vec{h}_{u(t)}^\nu(\lambda_t) = \vec{h}_1(\lambda_t) + u(t)\vec{h}_2(\lambda_t),$$

$$h_{u(t)}^\nu(\lambda_t) = \max_{u \in \mathbb{R}} h_u^\nu(\lambda_t),$$

$$(\nu, \lambda_t) \neq 0, \quad t \in [0, t_1].$$

Abnormal extremal trajectories

$$\nu = 0 \Rightarrow u(t) \equiv 0 \Rightarrow \theta \equiv 0, \quad x = t, \quad y \equiv 0$$



$$J = 0 = \text{min} \Rightarrow$$

\Rightarrow abnormal extremal trajectories optimal for $t \in [0, t_1]$

Unique trajectory from $q_0 = (0, 0, 0)$ to $(t_1, 0, 0) \in \partial \mathcal{A}_{q_0}(t_1)$.

Normal Hamiltonian system

$\nu = -1 \Rightarrow$ nonuniqueness of extremal trajectories

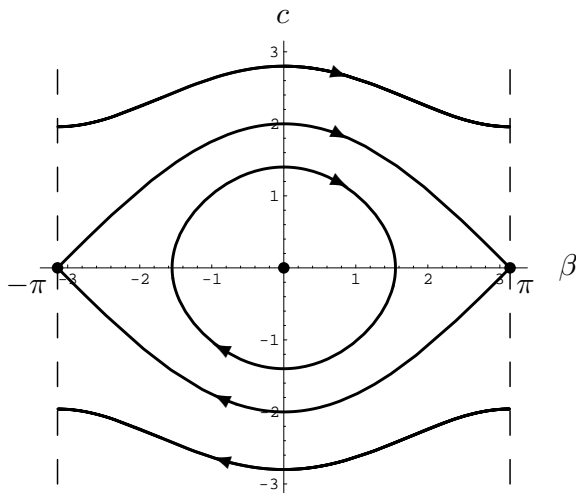
Hamiltonian system:

$$\begin{aligned} \dot{h}_1 &= -h_2 h_3, & \dot{x} &= \cos \theta \\ \dot{h}_2 &= h_3, & \dot{y} &= \sin \theta \\ \dot{h}_3 &= h_1 h_2, & \dot{\theta} &= h_2 \end{aligned}$$

$$r^2 = h_1^2 + h_3^2 \equiv \text{const} \Rightarrow h_1 = -r \cos \beta, \quad h_3 = -r \sin \beta$$

Equation of pendulum

$$\ddot{\beta} = -r \sin \beta \Leftrightarrow \begin{cases} \dot{\beta} = c, \\ \dot{c} = -r \sin \beta \end{cases}$$



Normal extremal trajectories

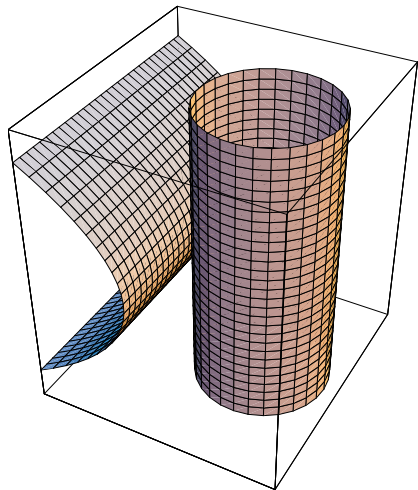
$$\begin{aligned}\ddot{\theta} &= -r \sin(\theta - \gamma), & r, \gamma &= \text{const}, \\ \dot{x} &= \cos \theta, \\ \dot{y} &= \sin \theta.\end{aligned}$$

Integrable in **Jacobi's functions**.

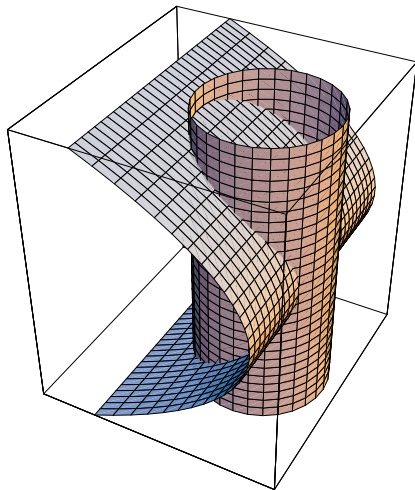
$\theta(t), x(t), y(t)$ parametrized by Jacobi's functions

$$\text{cn}(u, k), \quad \text{sn}(u, k), \quad \text{dn}(u, k), \quad E(u, k).$$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2$, $r^2 = h_1^2 + h_3^2$

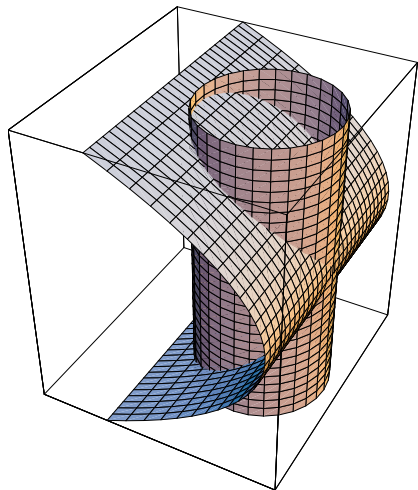


$$H = -r, r > 0$$

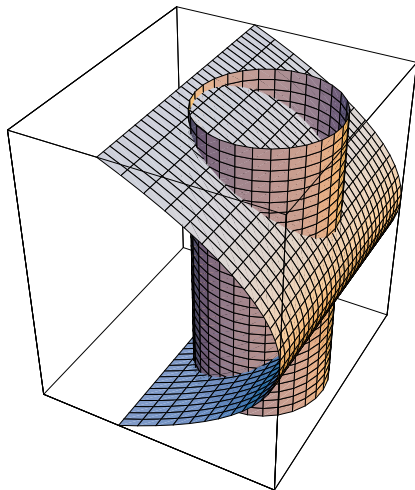


$$H \in (-r, r), r > 0$$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2$, $r^2 = h_1^2 + h_3^2$

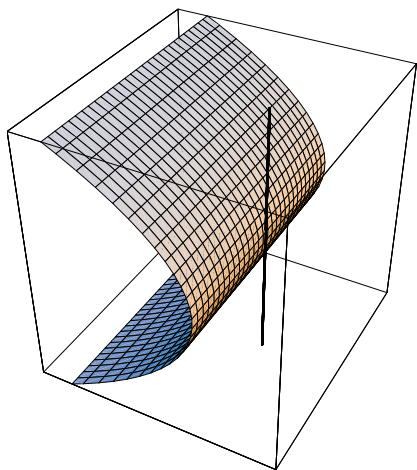


$$H = r > 0$$

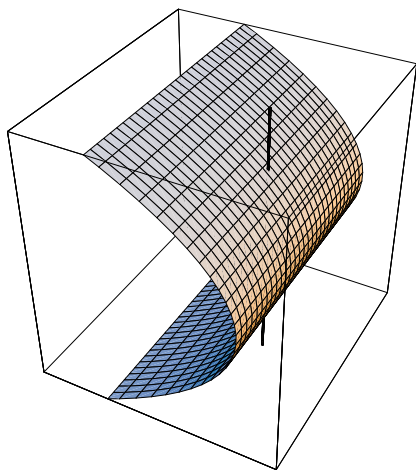


$$H > r > 0$$

Geometry of integrals $H = h_1 + \frac{1}{2}h_2^2$, $r^2 = h_1^2 + h_3^2$



$$H = r = 0$$



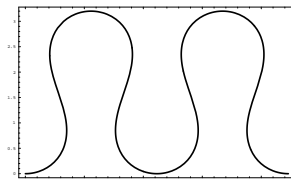
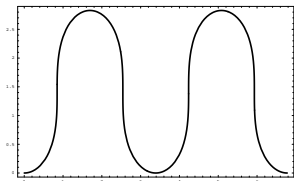
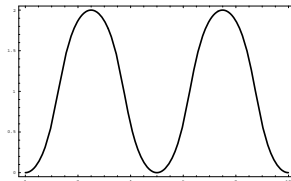
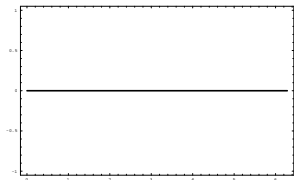
$$H > r = 0$$

Energy of pendulum

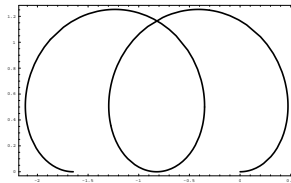
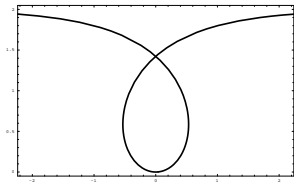
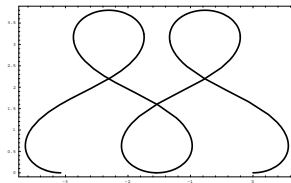
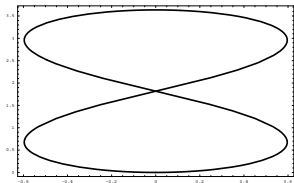
$$E = \frac{\dot{\theta}^2}{2} - r \cos(\theta - \gamma) \equiv \text{const} \in [-r, +\infty)$$

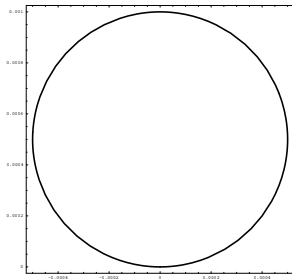
- $E = -r \neq 0 \Rightarrow$ straight lines
- $E \in (-r, r), r \neq 0 \Rightarrow$ inflectional elasticae
- $E = r \neq 0, \theta - \gamma = \pi \Rightarrow$ straight lines
- $E = r \neq 0, \theta - \gamma \neq \pi \Rightarrow$ critical elasticae
- $E > r \neq 0 \Rightarrow$ non-inflectional elasticae
- $r = 0 \Rightarrow$ straight lines and circles

Euler elasticae



Euler elasticae





Optimality of normal extremal trajectories

$q(t)$ **locally** optimal:

$$\exists \varepsilon > 0 \quad \forall \tilde{q}: \quad \|\tilde{q} - q\|_C < \varepsilon, \quad q(0) = \tilde{q}(0), \quad q(t_1) = \tilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\tilde{q})$$

Stable elastica $(x(t), y(t))$

$q(t)$ **globally** optimal:

$$\forall \tilde{q}: \quad q(0) = \tilde{q}(0), \quad q(t_1) = \tilde{q}(t_1) \quad \Rightarrow \quad J(q) \leq J(\tilde{q})$$

Elastica $(x(t), y(t))$ of minimal energy.

Theorem (Strong Legendre condition)

$$\frac{\partial^2}{\partial u^2} \Big|_{u(s)} h_u^{-1}(\lambda_s) < -\delta < 0 \quad \Rightarrow$$

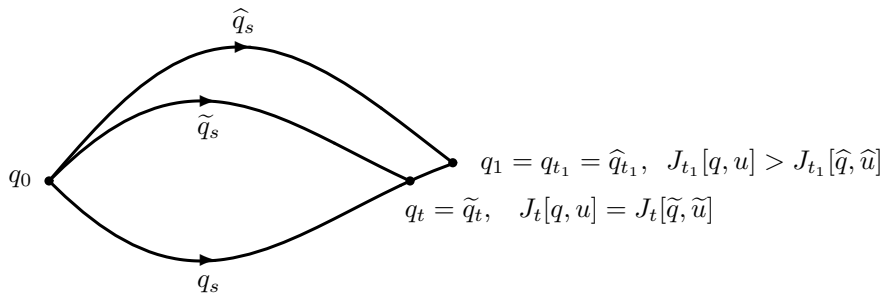
\Rightarrow *small arcs of normal extremal trajectories $q(s)$ are optimal.*

Cut time along $q(s)$:

$$t_{\text{cut}}(q) = \sup\{t > 0 \mid q(s), s \in [0, t], \text{ optimal}\}.$$

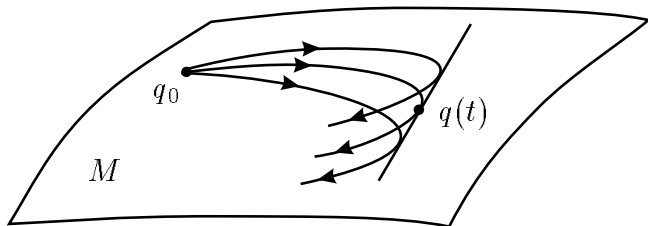
Reasons for loss of optimality: Maxwell point

Maxwell point q_t : $\exists \tilde{q}_s \neq q_s : q_t = \tilde{q}_t, J_t[q, u] = J_t[\tilde{q}, \tilde{u}]$



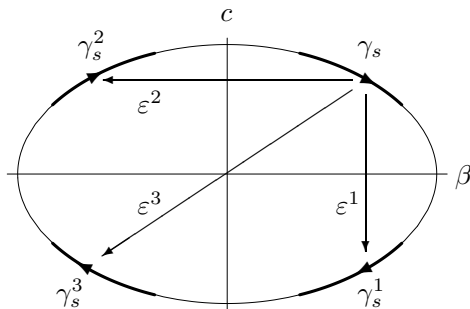
Reasons for loss of optimality: Conjugate point

Conjugate point: $q_t \in$ envelope of the family of extremal trajectories



$$t_{\text{cut}} \leq \min(t_{\text{Max}}, t_{\text{conj}})$$

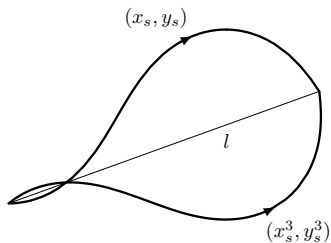
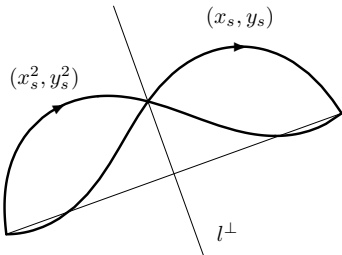
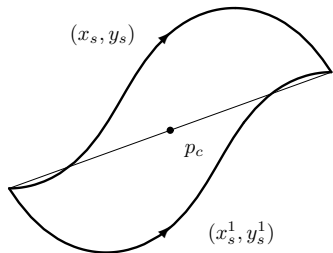
Reflections in the phase cylinder of pendulum $\ddot{\beta} = -r \sin \beta$



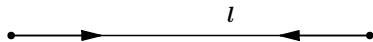
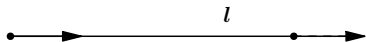
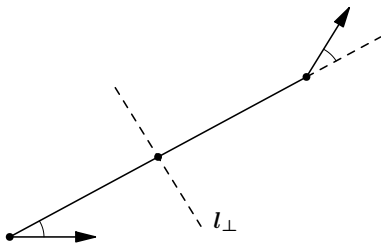
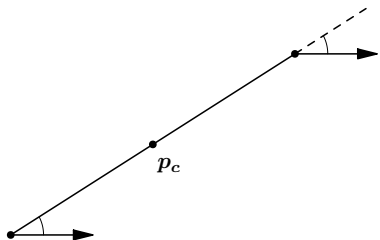
Dihedral group $D_2 = \{\text{Id}, \varepsilon^1, \varepsilon^2, \varepsilon^3\}$

	ε^1	ε^2	ε^3
ε^1	ld	ε^3	ε^2
ε^2	ε^3	ld	ε^1
ε^3	ε^2	ε^1	ld

Action of reflections $\varepsilon^1, \varepsilon^2, \varepsilon^3$ on elasticae



Fixed points of reflections $\varepsilon^1, \varepsilon^2, \varepsilon^3$



Maxwell points corresponding to reflections

Fixed points of reflections $\varepsilon^i \Rightarrow$ Maxwell times:

$$t = t_{\varepsilon^i}^n, \quad i = 1, 2, \quad n = 1, 2, \dots$$

$T =$ period of pendulum \Rightarrow

$$t_{\varepsilon^1}^n = nT, \quad \left(n - \frac{1}{2}\right) T < t_{\varepsilon^2}^n < \left(n + \frac{1}{2}\right) T.$$

Upper bound of cut time:

$$t_{\text{cut}} \leq \min(t_{\varepsilon^1}^1, t_{\varepsilon^2}^1) \leq T.$$

Conjugate points

Exponential mapping

$$\text{Exp}_t : T_{q_0}^* M \rightarrow M, \quad \lambda_0 \mapsto q = q(t) = \pi \circ e^{t\vec{h}}(\lambda_0)$$

q — conjugate point $\Leftrightarrow q$ — critical value of Exp_t

$$\text{Exp}_t(h_1, h_2, h_3) = (x, y, \theta)$$

$$\frac{\partial(x, y, \theta)}{\partial(h_1, h_2, h_3)} = 0$$

Local optimality of normal extremal trajectories

$q(t) = (x(t), y(t), \theta(t))$ normal extremal trajectory

Theorem (Jacobi condition)

- *no conjugate points at $(0, t_1]$ \Rightarrow $q(t)$ is locally optimal;*
- *$(0, t_1)$ contains conjugate points \Rightarrow $q(t)$ is not locally optimal.*

Local optimality is lost at the **first conjugate point** $t_{\text{conj}}^1 \in (0, +\infty]$

- No inflection points \Rightarrow no conjugate points
- Inflectional case $\Rightarrow t_{\text{conj}}^1 \in [t_{\varepsilon^1}^1, t_{\varepsilon^2}^1] \subset [\frac{1}{2}T, \frac{3}{2}T]$

$(x(s), y(s))$ stable $\Leftrightarrow q(s) = (x(s), y(s), \theta(s))$ locally optimal

- $t_1 < t_{\text{conj}}^1 \Rightarrow$ stability
- $t_1 > t_{\text{conj}}^1 \Rightarrow$ instability
- straight lines, circles, non-inflectional elasticae are **stable**

Stability of inflectional elasticae

Loss of stability at the **first conjugate point**

- $t_1 \leq \frac{1}{2}T \Rightarrow$ stability
- $t_1 \geq \frac{3}{2}T \Rightarrow$ instability

In particular:

- no inflection points \Rightarrow stability
- 1 or 2 inflection points \Rightarrow stability or instability
- 3 inflection points \Rightarrow instability

Global optimality of elasticae

$$q_1 \in \mathcal{A}_{q_0}(t_1), \quad \text{optimal } q(t) = ?$$

$$q(t) = \text{Exp}_t(\lambda) \text{ optimal for } t \in [0, t_1] \quad \Rightarrow \quad t_1 \leq \min(t_{\varepsilon_1}^1(\lambda), t_{\varepsilon_2}^1(\lambda))$$

$$N' = \{\lambda \in T_{q_0}^* M \mid t_1 \leq \min(t_{\varepsilon_1}^1(\lambda), t_{\varepsilon_2}^1(\lambda))\}$$

$\text{Exp}_{t_1} : N' \rightarrow \mathcal{A}_{q_0}(t_1)$ surjective, with singularities and multiple points

\exists open dense $\tilde{N} \subset N'$, $\tilde{M} \subset \mathcal{A}_{q_0}(t_1)$ such that

$$\text{Exp}_{t_1} : \tilde{N} \rightarrow \tilde{M} \text{ double covering}$$

Global structure of exponential mapping

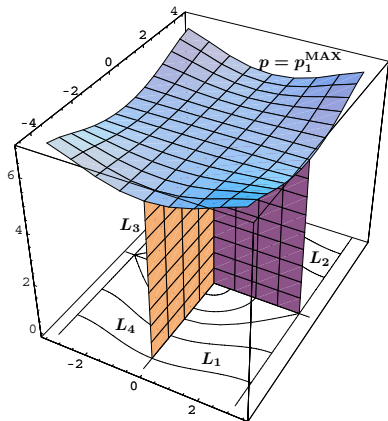


Figure: $\tilde{N} = \cup_{i=1}^4 L_i$

Exp_{t_1}
 \longrightarrow

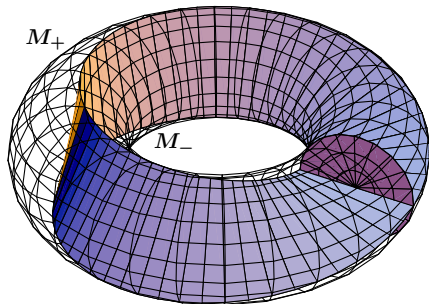


Figure: $\tilde{M} = M_+ \cup M_-$

$\text{Exp}_{t_1} : L_1, L_3 \rightarrow M_+$ diffeo,

$\text{Exp}_{t_1} : L_2, L_4 \rightarrow M_-$ diffeo

Competing elasticae

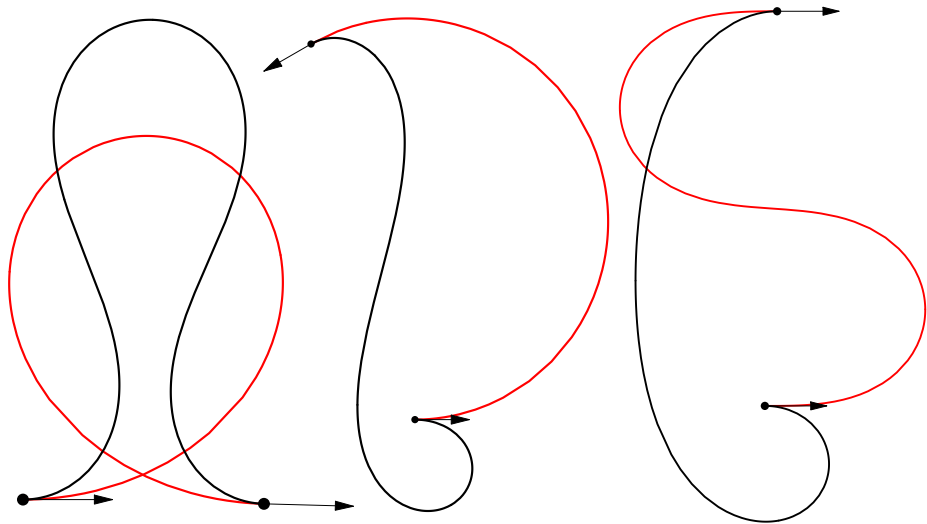
$$q^1(t) = \text{Exp}_t(\lambda^1)$$
$$\lambda^1 \in L_1$$

$$q_1 \in M_+$$

$$q^2(t) = \text{Exp}_t(\lambda^2)$$
$$\lambda^2 \in L_3$$

$$? : J[q^1] \leq J[q^2]$$

Competing elasticae



Questions and perspectives

- Cut time $t_{\text{cut}} = ?$
- Optimal synthesis in Euler's elastic problem
- Nilpotent $(2, 3, 5)$ sub-Riemannian problem
- The ball-plate problem
- Sub-Riemannian problem on $\mathbb{E}(2)$

Conclusion: Euler's elastic problem

- Optimal control problem
- Extremal trajectories
- Local and global optimality of extremal trajectories
- Stability of Euler elasticae

[1] Yu. L. Sachkov, Maxwell strata in Euler's elastic problem, arXiv:0705.0614 [math.OC], 3 May 2007, *submitted*.

[2] Yu. L. Sachkov, Conjugate points in Euler's elastic problem, arXiv:0705.1003 [math.OC], 7 May 2007, *submitted*.

[3] Yu. L. Sachkov, Optimality of Euler elasticae, Doklady Mathematics, *to appear*.