Symmetries and Maxwell points in the plate-ball problem and other invariant optimal control problems on Lie groups governed by the pendulum

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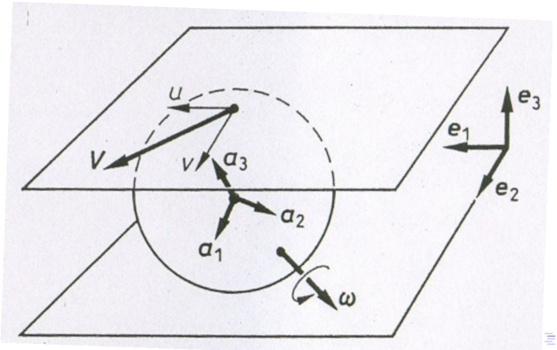
The plate-ball problem

Rolling of sphere on plane without slipping or twisting

Given: $A, B \in \mathbb{R}^2$, frames (a_1, a_2, a_3), (b_1, b_2, b_3) in $\mathbb{R}^3$.

Find: $\gamma(t) \in \mathbb{R}^2$, $t \in [0, t_1]$, — the shortest curve s.t.:
$\gamma(0) = A$, $\gamma(t_1) = B$,
by rolling along $\gamma(t)$, orientation of the sphere transfers from (a_1, a_2, a_3) to (b_1, b_2, b_3).



State and control variables

- Contact point $(x, y) \in \mathbb{R}^2$
- Orientation of sphere $R : a_i \mapsto e_i, \ i = 1, 2, 3, \qquad R \in \mathrm{SO}(3)$
- State of the system $Q = (x, y, R) \in \mathbb{R}^2 \times \mathrm{SO}(3) = M$
- Boundary conditions $Q(0) = Q_0, \ Q(t_1) = Q_1$
- Controls $u_1 = u/2, \ u_2 = v/2$
- Cost functional
$$l(\gamma) = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2}\, dt = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min$$

Control system

$$\dot{x} = u_1, \qquad \dot{y} = u_2, \qquad (x, y) \in \mathbb{R}^2, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$\dot{R} = R\Omega, \qquad R \in SO(3), \qquad \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix},$$

$$\omega = \begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} \text{ angular velocity vector.}$$

No twisting $\Rightarrow$ $\omega_3 = 0$.
No slipping $\Rightarrow$ $\omega_1 = u_2$, $\omega_2 = -u_1$.

$$\Omega = \begin{pmatrix} 0 & 0 & -u_1 \\ 0 & 0 & -u_2 \\ u_1 & u_2 & 0 \end{pmatrix}$$

History of the problem

1894 H. Hertz: rolling sphere as a nonholonomic mechanical system.

1983 J.M. Hammersley: statement of the plate-ball problem.

1986 A.M. Arthur, G.R.Walsh: integrability of Hamiltonian system of PMP in quadratures.

1990 Z. Li, E. Canny: complete controllability of the control system.

1993 V. Jurdjevic:

- projections of extremal curves $(x(t), y(t))$ — Euler elasticae,
- description of qualitative types of extremal trajectories,
- quadratures for evolution of Euler angles along extremal trajectories.

New results

- Parameterization of extremal trajectories
- Continuous and discrete symmetries
- Fixed points of symmetries (Maxwell points)
- Necessary optimality conditions
- Global structure of the exponential mapping
- Asymptotics of extremal trajectories and limit behavior of Maxwell points for sphere rolling along sinusoids of small amplitude (Next talk by Alexey Mashtakov)

Existence of solutions

- Left-invariant sub-Riemannian problem:

$$\dot{Q} = u_1 X_1(Q) + u_2 X_2(Q), \qquad (u_1, u_2) \in \mathbb{R}^2,$$
$$Q(0) = Q_0, \qquad Q(t_1) = Q_1, \qquad Q \in M = \mathbb{R}^2 \times \mathrm{SO}(3),$$
$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

- Complete controllability by Rashevskii-Chow theorem:

$$\mathrm{span}_Q(X_1, X_2, X_3, X_4, X_5) = T_Q M \quad \forall\ Q \in M,$$
$$X_3 = [X_1, X_2], \qquad X_4 = [X_1, X_3], \qquad X_5 = [X_2, X_3].$$

- Filippov's theorem: $\forall Q_0, Q_1 \in M$ optimal trajectory exists.
- $Q_0 = (0, 0, \mathrm{Id}) \in \mathbb{R}^2 \times \mathrm{SO}(3)$.

Pontryagin maximum principle

- Abnormal extremal trajectories: rolling of sphere along straight lines.
- Normal extremals:

$$\dot\theta = c, \quad \dot c = -r\sin\theta, \quad \dot\alpha = \dot r = 0, \tag{1}$$

$$\dot x = \cos(\theta + \alpha), \quad \dot y = \sin(\theta + \alpha), \tag{2}$$

$$\dot R = R(\sin(\theta + \alpha)A_1 - \cos(\theta + \alpha)A_2),$$

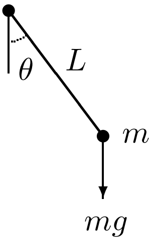
$$A_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix},$$

$$A_3 = [A_1, A_2] = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

(1) mathematical pendulum,

(2) Euler elasticae.

Mathematical pendulum $\dot{\theta} = c$, $\dot{c} = -r \sin \theta$

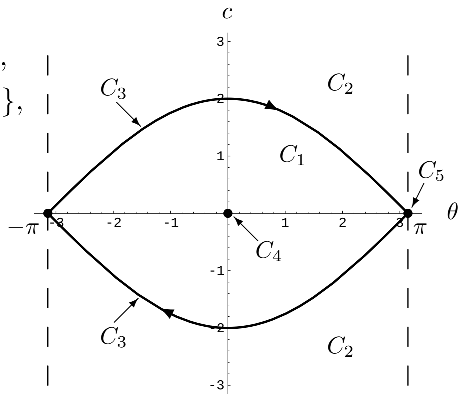


- $\lambda = (\theta, c, r) \in C = \{\theta \in S^1,\ c \in \mathbb{R},\ r \geq 0\}$,
- Energy $E = c^2/2 - r \cos \theta = \mathrm{const} \in [-r, +\infty)$,
- $r = g/L \geq 0$.

Stratificaion of the phase cylinder C of pendulum

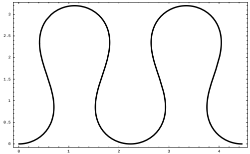
$$C = \cup_{i=1}^{7} C_i, \qquad C_i \cap C_j = \emptyset, \quad i \neq j$$

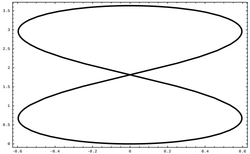
$$C_1 = \{\lambda \in C \mid E \in (-r, r),\ r > 0\},$$
$$C_2 = \{\lambda \in C \mid E \in (r, +\infty),\ r > 0\},$$
$$C_3 = \{\lambda \in C \mid E = r > 0,\ c \neq 0\},$$
$$C_4 = \{\lambda \in C \mid E = -r,\ r > 0\},$$
$$C_5 = \{\lambda \in C \mid E = r > 0,\ c = 0\},$$
$$C_6 = \{\lambda \in C \mid r = 0,\ c \neq 0\},$$
$$C_7 = \{\lambda \in C \mid r = 0,\ c = 0\}.$$

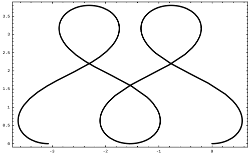


Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

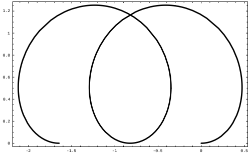
C_1 (oscillations of pendulum): inflectional elasticae





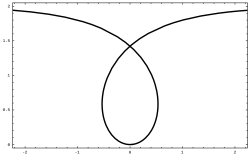


C_2 (rotations of pendulum): non-inflectional elasticae



Euler elasticae $\dot{x} = \cos(\theta + \alpha)$, $\dot{y} = \sin(\theta + \alpha)$

C_3 (separatrix motions of penulum): critical elasticae



C_4, C_5, C_7 (equilibria of pendulum): straight lines

C_6 (uniform rotation of pendulum under zero gravity): circles

Integration of normal Hamiltonian system of PMP

$$\dot{\theta} = c, \qquad \dot{c} = -r \sin\theta, \qquad \dot{x} = \cos(\theta + \alpha), \qquad \dot{y} = \sin(\theta + \alpha),$$
$$\dot{R} = R(\sin(\theta + \alpha) A_1 - \cos(\theta + \alpha) A_2).$$

- θ_t, c_t, x_t, y_t: Jacobi's functions cn, sn, dn, E,

$$\operatorname{cn}(u,k) = \cos(\operatorname{am}(u,k)), \qquad \operatorname{sn}(u,k) = \sin(\operatorname{am}(u,k)),$$
$$\varphi = \operatorname{am}(u,k) \iff u = \int_0^{\varphi} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}} = F(\varphi, k).$$

- $R(t) = e^{(\alpha - \varphi_3^0) A_3} e^{-\varphi_2^0 A_2} e^{\varphi_1(t) A_3} e^{\varphi_2(t) A_2} e^{(\varphi_3(t) - \alpha) A_3}$,
  $\varphi_i(t)$: Jacobi's functions + elliptic integral of the 3-rd kind

$$\Pi(n, u, k) = \int_0^{u} \frac{dt}{(1 - n \sin^2 t)\sqrt{1 - k^2 \sin^2 t}}.$$

Parameterization of trajectories of oscillating pendulum and inflectional Euler elasticae

(φ, k) — coordinates rectifying the flow of pendulum,

$$\varphi_t = \varphi + t,$$

$$\sin(\theta_t/2) = k \operatorname{sn}(\sqrt{r}\varphi_t, k), \qquad \cos(\theta_t/2) = \operatorname{dn}(\sqrt{r}\varphi_t, k),$$

$$c_t = 2k\sqrt{r} \operatorname{cn}(\sqrt{r}\varphi_t, k),$$

$$x_t = \bar{x}_t \cos\alpha - \bar{y}_t \sin\alpha, \qquad y_t = \bar{x}_t \sin\alpha + \bar{y}_t \cos\alpha,$$

$$\bar{x}_t = (2(\mathrm{E}(\sqrt{r}\varphi_t, k) - \mathrm{E}(\sqrt{r}\varphi, k)) - \sqrt{r}t)/\sqrt{r},$$

$$\bar{y}_t = 2k(\operatorname{cn}(\sqrt{r}\varphi, k) - \operatorname{cn}(\sqrt{r}\varphi_t, k))/\sqrt{r},$$

Parameterization of the matrix of rotation for the case of oscillating pendulum

$$\cos\varphi_2(t) = c_t/\sqrt{M}, \qquad \sin\varphi_2(t) = \pm\sqrt{M - c_t^2}/\sqrt{M},$$

$$\cos\varphi_3(t) = \mp\sin\theta_t/\sqrt{M - c_t^2},$$

$$\sin\varphi_3(t) = \pm(r - \cos\theta_t)/\sqrt{M - c_t^2},$$

$$\varphi_1(t) = \frac{\sqrt{M}}{2}t + \frac{\sqrt{M}(1+r)}{2\sqrt{r}(1-r)}(\Pi(l, \mathrm{am}(\sqrt{r}\varphi_t, k), k) - \Pi(l, \mathrm{am}(\sqrt{r}\varphi, k), k)),$$

$$M = 1 + r^2 + 2E, \qquad l = -\frac{4k^2 r}{(1-r)^2}.$$

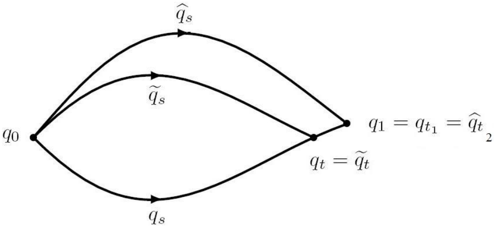
Optimality of extremal trajectories

- Short arcs of extremal trajectories $Q(s)$ are optimal
- Cut time along $Q(s)$:

$$t_{\mathrm{cut}} = \sup\{t > 0 \mid Q(s),\ s \in [0, t],\ \ \text{is optimal}\ \}.$$

- Maxwell time:

$$\exists\ \tilde{Q}(s) \not\equiv Q(s), \quad Q(0) = \tilde{Q}(0) = Q_0,$$
$$Q(t) = \tilde{Q}(t)\ \text{Maxwell point},$$
$$t = t_{\mathrm{Max}}\ \text{Maxwell time}.$$



- Upper bound on cut time: $t_{\mathrm{cut}} \leq t_{\mathrm{Max}}$.

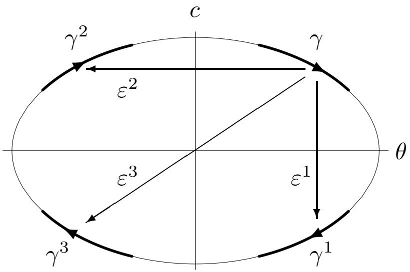
Rotations Φ^β, $\beta \in S^1$

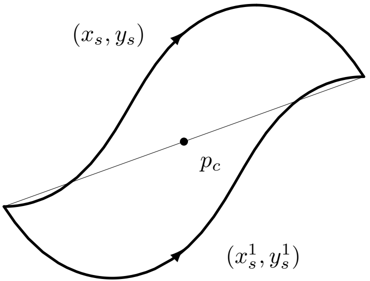
$$(\theta, c, r, \alpha) \mapsto (\theta, c, r, \alpha + \beta),$$

$$\begin{pmatrix} x_s \\ y_s \end{pmatrix} \mapsto \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} x_s \\ y_s \end{pmatrix},$$

$$R_s \mapsto e^{\beta A_3} R_s e^{-\beta A_3}.$$

Reflections ε^i



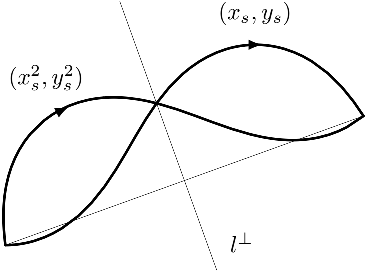


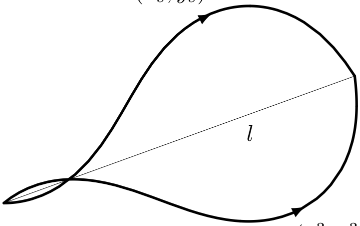
ε^1: $(\theta_s, c_s) \mapsto (\theta_{t-s}, -c_{t-s}),\ s \in [0, t]$

$(x_s, y_s) \mapsto (x_s^1, y_s^1) = (x_{t-s} - x_t, y_{t-s} - y_t)$

$R_s \mapsto (R_t)^{-1} R_{t-s}$

Reflections ε^i





ε^2: $(\theta_s, c_s) \mapsto (-\theta_{t-s}, c_{t-s})$, $s \in [0, t]$
$(x_s, y_s) \mapsto (x_s^2, y_s^2) = (x_{t-s} - x_t, y_t - y_{t-s})$
$R_s \mapsto I_2(R_t)^{-1} R_{t-s} I_2$, $I_2 = e^{\pi A_2}$.

ε^3: $(\theta_s, c_s) \mapsto (-\theta_s, -c_s)$, $s \in [0, t]$
$(x_s, y_s) \mapsto (x_s^3, y_s^3) = (x_s, -y_s)$
$R_s \mapsto I_2 R_s I_2$.

Exponential mapping and its symmetries

- Group of symmetries
  $G = \langle \Phi^\beta, \varepsilon^1, \varepsilon^2, \varepsilon^3 \rangle = \{\Phi^\beta,\ \Phi^\beta \circ \varepsilon^i \mid \beta \in S^1,\ i = 1, 2, 3\}$
- Exponential mapping

$$\operatorname{Exp}(\lambda, s) = Q_s = (x_s, y_s, R_s) \in M = \mathbb{R}^2 \times \mathrm{SO}(3),$$
$$\lambda = (\theta, c, \alpha, r) \in C, \qquad s > 0.$$

- Symmetries of exponential mapping

$$\begin{tikzcd} C \times \mathbb{R}_+ \arrow[r, "\operatorname{Exp}"] \arrow[d, "\varepsilon^i \circ \Phi^\beta"] & M \arrow[d, "\varepsilon^i \circ \Phi^\beta"] \\ C \times \mathbb{R}_+ \arrow[r, "\operatorname{Exp}"] & M \end{tikzcd} \qquad \begin{tikzcd} (\lambda, t) \arrow[r, mapsto, "\operatorname{Exp}"] \arrow[d, mapsto, "\varepsilon^i \circ \Phi^\beta"] & Q_t \arrow[d, mapsto, "\varepsilon^i \circ \Phi^\beta"] \\ (\lambda^{i,\beta}, t) \arrow[r, mapsto, "\operatorname{Exp}"] & Q_t^{i,\beta} \end{tikzcd}$$

Maxwell sets corresponding to reflections

- $\mathrm{MAX}^i = \{(\lambda, t) \mid \exists \beta \in S^1 \ : \ \lambda^{i,\beta} \neq \lambda, \ Q_t = Q_t^{i,\beta}\}$, $i = 1, 2, 3$.

- Necessary optimality conditions:

$$(\lambda, t) \in \mathrm{MAX}^i \quad \Rightarrow \quad \begin{array}{l} Q_s = \mathrm{Exp}(\lambda, s) \text{ not optimal for } s > t, \\ t_{\mathrm{cut}}(\lambda) \leq t. \end{array}$$

Representation of rotations in $\mathbb{R}^3$ by quaternions

- $\mathbb{H} = \{q = q_0 + iq_1 + jq_2 + kq_3 | q_0, \ldots, q_3 \in \mathbb{R}\}$
- $S^3 = \{q \in \mathbb{H} | |q|^2 = q_0^2 + q_1^2 + q_2^2 + q_3^2 = 1\}$
- $I = \{q \in \mathbb{H} | \mathrm{Re}\, q = q_0 = 0\}$
- $q \in S^3 \Rightarrow R_q(a) = qaq^{-1}, \quad a \in I, \quad R_q \in \mathrm{SO}(3) \cong \mathrm{SO}(I)$
- lift of the system $\dot{R} = R\Omega$ from SO(3) to S^3:

$$\begin{cases} \dot{q}_0 = \frac{1}{2}(q_2 u_1 - q_1 u_2), \\ \dot{q}_1 = \frac{1}{2}(q_3 u_1 + q_0 u_2), \\ \dot{q}_2 = \frac{1}{2}(-q_0 u_1 + q_3 u_2), \\ \dot{q}_3 = \frac{1}{2}(-q_1 u_1 - q_2 u_2), \end{cases} \qquad q \in S^3, \quad (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = 1.$$

Necessary optimality conditions in terms of MAX^1

Theorem

Let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$, $t > 0$ *satisfy the conditions:*

(1) $q_3(t) = 0$,

(2) *elastica* $\{(x_s, y_s) \mid s \in [0, t]\}$ *is nondegenerate and not centered at inflection point.*

Then $(\lambda, t) \in \mathrm{MAX}^1$, *thus for any* $t_1 > t$ *the trajectory* Q_s, $s \in [0, t_1]$, *is not optimal.*

$$q_3(t) = 0 \iff \text{axis of rotation } (q_1(t), q_2(t), q_3(t)) \parallel \mathbb{R}^2$$

Necessary optimality conditions in terms of MAX^2

Theorem

Let $Q_s = (x_s, y_s, R_s) = \mathrm{Exp}(\lambda, s)$, $t > 0$ *satisfy the conditions:*

(1) $(xq_1 + yq_2)(t) = 0$,

(2) *elastica* $\{(x_s, y_s) \mid s \in [0, t]\}$ *is nondegenerate and not centered at vertex.*

Then $(\lambda, t) \in \mathrm{MAX}^2$, *thus for any* $t_1 > t$ *the trajectory* Q_s, $s \in [0, t_1]$, *is not optimal.*

$$(xq_1 + yq_2)(t) = 0 \iff (q_1(t), q_2(t), q_3(t)) \perp (x(t), y(t), 0)$$

Global structure of exponential mapping

- $\mathrm{Exp} : N \to M$,
  $N = C \times \mathbb{R}_+$
  $= \{(\theta, c, \alpha, r, t) \mid \theta \in S^1,\ c \in \mathbb{R},\ \alpha \in S^1,\ r \geq 0,\ t > 0\}$,
  $M = \mathbb{R}^2 \times \mathrm{SO}(3)$

- $\forall Q_1 \in M \setminus Q_0 \ \ \exists (\lambda, t) \in N$ such that $Q_s = \mathrm{Exp}(\lambda, s)$ optimal,
  $Q_1 = \mathrm{Exp}(\lambda, t)$
  $t \leq t_{\mathrm{cut}}(\lambda) \leq t^1_{\mathrm{Max}} = \inf\{s \mid (\lambda, s) \in \mathrm{MAX}^1 \cup \mathrm{MAX}^2\}$
  $(\lambda, t) \in \widehat{N} = \{(\lambda, s) \in N \mid \lambda \in C,\ 0 < s \leq t^1_{\mathrm{Max}}\}$
  $\mathrm{Exp} : \widehat{N} \to M \setminus Q_0$ surjective

Global structure of exponential mapping

- Decomposition in the preimage of Exp:
  $\widehat{N} \supset \cup_{i=1}^4 N_i$, $\mathrm{cl}(\cup_{i=1}^4 N_i) \supset \widehat{N}$,
  $N_i = \{(\lambda, t) \in D_i \mid 0 < t < t^1_{\mathrm{Max}}(\lambda)\}$,
  $D_i = \{(\lambda, t) \in N \mid \operatorname{sgn} c_{t/2} = \pm 1, \ \operatorname{sgn} \sin \theta_{t/2} = \pm 1\}$.

- Decomposition in the image of Exp:
  $M \supset M_1 \cup M_2$, $\mathrm{cl}(M_1 \cup M_2) = M$,
  $M_i = \{(x, y, Q) \in M \mid q_3 > 0, \ \operatorname{sgn}(xq_1 + yq_2) = \pm 1\}$.

- Conjecture:
  $\mathrm{Exp} \ : \ N_1, \ N_3 \to M_1$ are diffeomorphisms,
  $\mathrm{Exp} \ : \ N_2, \ N_4 \to M_2$ are diffeomorphisms.

Steps required to prove the conjecture

- N_i, M_i connected, open (proved: diffeomorphic to $\mathbb{R}^4 \times S^1$),
- $N_i/\{\Phi^\beta \mid \beta \in S^1\}$, $M_i/\{\Phi^\beta \mid \beta \in S^1\}$ simply connected (proved : diffeomorphic to $\mathbb{R}^4$),
- $\mathrm{Exp}(N_1), \mathrm{Exp}(N_3) \subset M_1$, $\mathrm{Exp}(N_2), \mathrm{Exp}(N_4) \subset M_2$ (proved),
- $\mathrm{Exp}(\partial N_i) \subset \partial M_1 \cup \partial M_2$ (proved),
- $\mathrm{Exp} : N_1, \; N_3 \to M_1$, $\mathrm{Exp} : N_2, \; N_4 \to M_2$ proper (partially proved),
- $\mathrm{Exp}|_{N_i}$ nondegenerate (numerical evidence).

Algorithm for solution to the problem (modulo the conjecture)

- $Q_1 \in M_1 \cup M_2 \quad \Rightarrow \quad$ optimal trajectory $Q_s = ?$
- $Q_1 \in M_1 \quad \Rightarrow$

  $\exists!(\lambda_1, t_1) \in N_1$ such that $\mathrm{Exp}(\lambda_1, t_1) = Q_1$,
  $\exists!(\lambda_3, t_3) \in N_3$ such that $\mathrm{Exp}(\lambda_3, t_3) = Q_1$,

  $$\begin{aligned} t_1 < t_3 \quad &\Rightarrow \quad Q_s^1 = \mathrm{Exp}(\lambda_1, s) \text{ optimal},\\ t_1 > t_3 \quad &\Rightarrow \quad Q_s^3 = \mathrm{Exp}(\lambda_3, s) \text{ optimal},\\ t_1 = t_3 \quad &\Rightarrow \quad Q_s^1,\ Q_s^3 \text{ optimal}. \end{aligned}$$
- $Q_1 \in M_2 \quad \Rightarrow \quad$ similarly for $(\lambda_2, t_2) \in N_2$, $(\lambda_4, t_4) \in N_4$.

Results and plans for the plate-ball problem

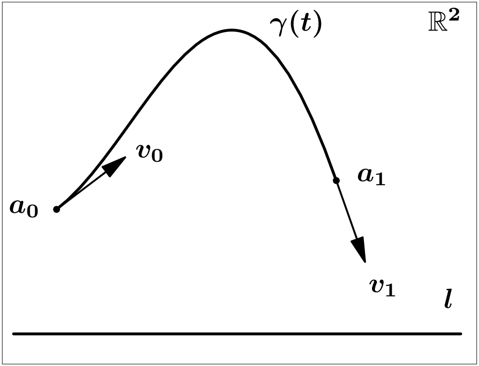
- parameterization of extremal trajectories,
- symmetries and Maxwell points,
- upper bound on cut time,
- global structure of the exponential mapping,
- software for numerical solution to the plate-ball problem.

The zoo of invariant optimal control problems on Lie groups governed by the pendulum

$$\ddot{\theta} = -r \sin(\theta - \alpha)$$

- SR problem on the group of motions of a plane
- Euler's elastic problem
- SR problem on the Engel group
- nilpotent SR problem with the growth vector $(2, 3, 5)$
- the plate-ball problem

Euler's elastic problem



Given: $l > 0$, $a_0, a_1 \in \mathbb{R}^2$, $v_0 \in T_{a_0}\mathbb{R}^2$, $v_1 \in T_{a_1}\mathbb{R}^2$, $|v_0| = |v_1| = 1$.
Find: $\gamma(t)$, $\quad t \in [0, t_1]$:
$\gamma(0) = a_0$, $\gamma(t_1) = a_1$, $\dot\gamma(0) = v_0$, $\dot\gamma(t_1) = v_1$.
$|\dot\gamma(t)| \equiv 1 \quad \Rightarrow \quad t_1 = l$
Elastic energy $J = \dfrac{1}{2}\displaystyle\int_0^{t_1} k^2\, dt \to \min$, $\quad k(t)$ — curvature of $\gamma(t)$.

Results for Euler's elastic problem

- Parameterization of extremal trajectories,
- Symmetries, Maxwell strata, Maxwell time,
- Bound of conjugate time,
- Global structure of exponential mapping,
- Software for computation of optimal elasticae.

Global structure of exponential mapping

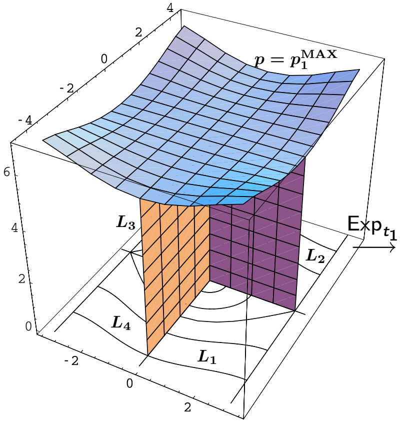


Figure: $\widetilde{N} = \cup_{i=1}^{4} L_i$

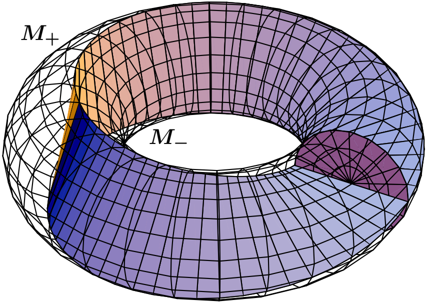
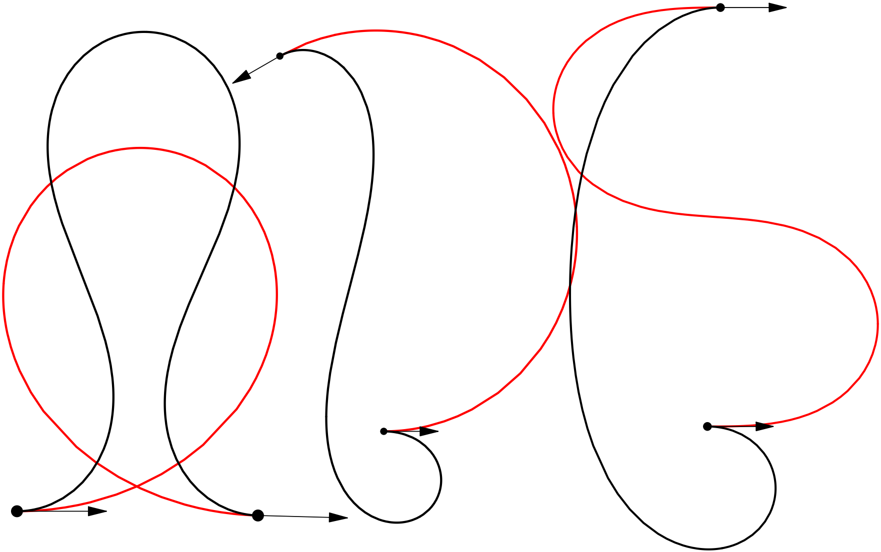


Figure: $\widetilde{M} = M_+ \cup M_-$

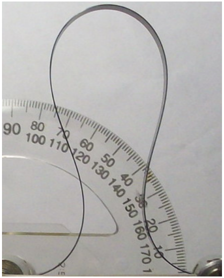
$\mathrm{Exp}_{t_1} \; : \; L_1, \; L_3 \to M_+$ diffeo, $\qquad \mathrm{Exp}_{t_1} \; : \; L_2, \; L_4 \to M_-$ diffeo

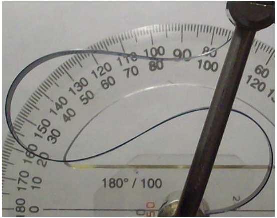
Competing elasticae

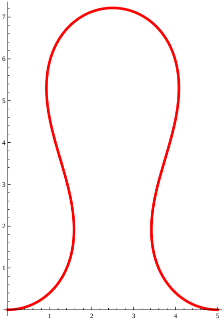


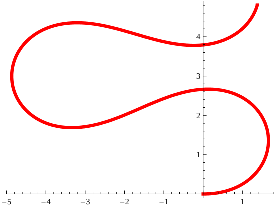
Movies with globally optimal elasticae . . .

One-parameter family of elasticae with loss of optimality

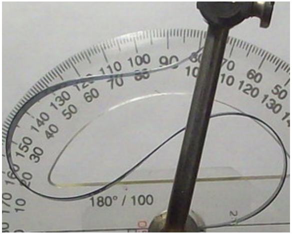


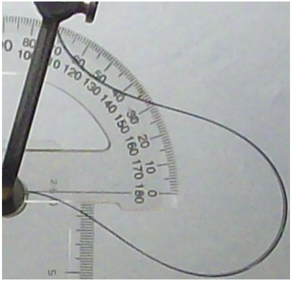


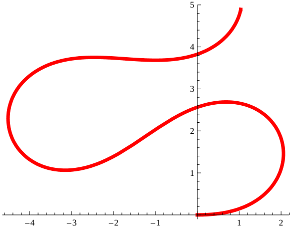


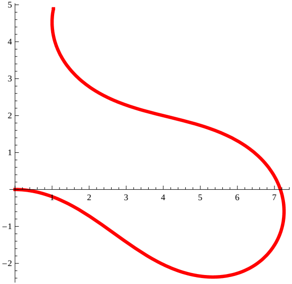


One-parameter family of elasticae with loss of optimality

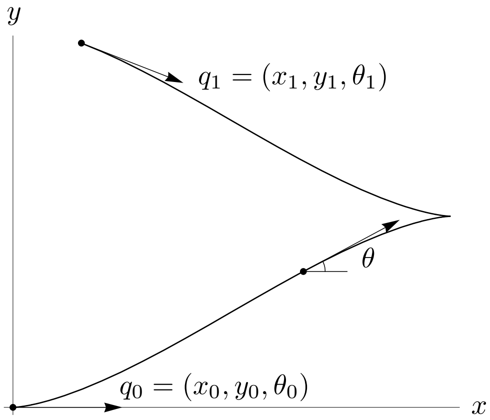








SR problem on SE(2)



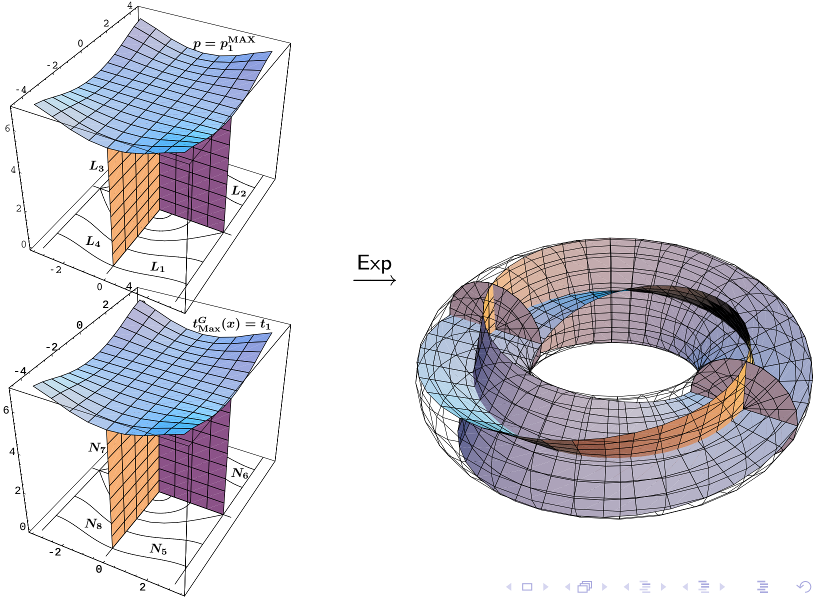
$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \beta^2\ \dot{\theta}^2}\, dt \to \min \qquad (\beta = 1)$$

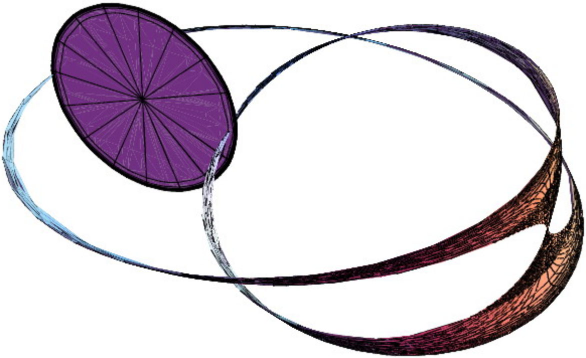
Results for SR problem on SE(2)

- Parameterization of extremal trajectories,
- Symmetries, Maxwell strata, Maxwell time,
- Bound of conjugate time,
- Global structure of exponential mapping,
- Global structure of cut locus and spheres,
- Software for computation of optimal trajectories,
- Application to reconstruction of images.

Global structure of exponential mapping

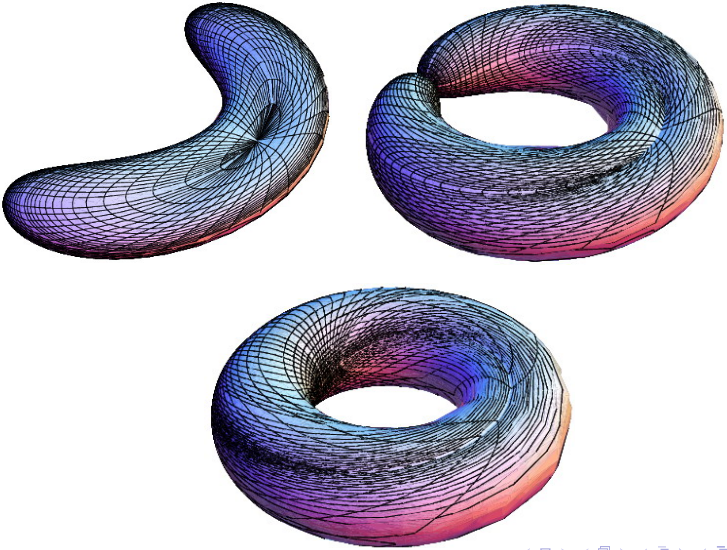


Cut locus: global view

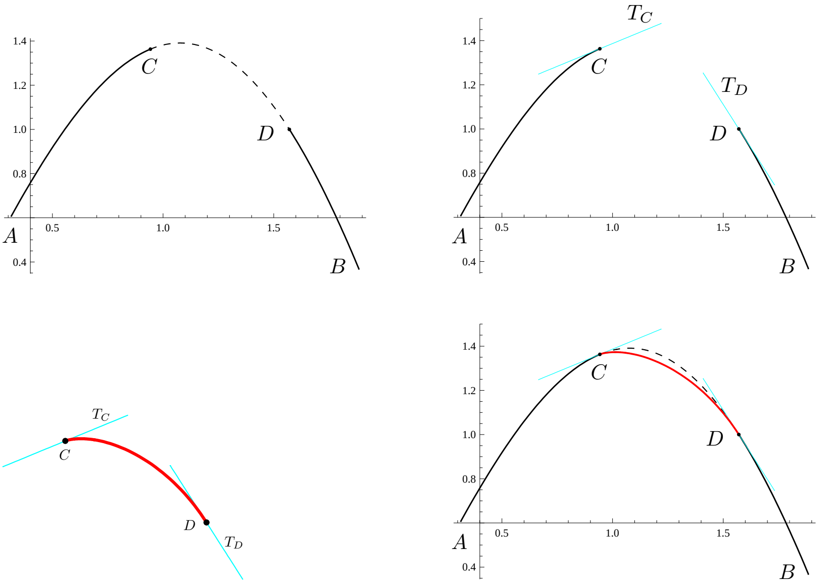


Global structure of sub-Riemannian spheres:

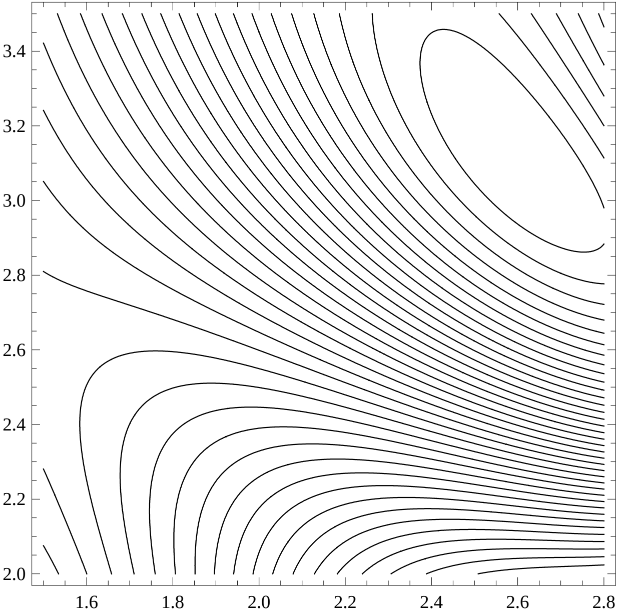
$$R < \pi, \qquad R = \pi, \qquad R > \pi$$



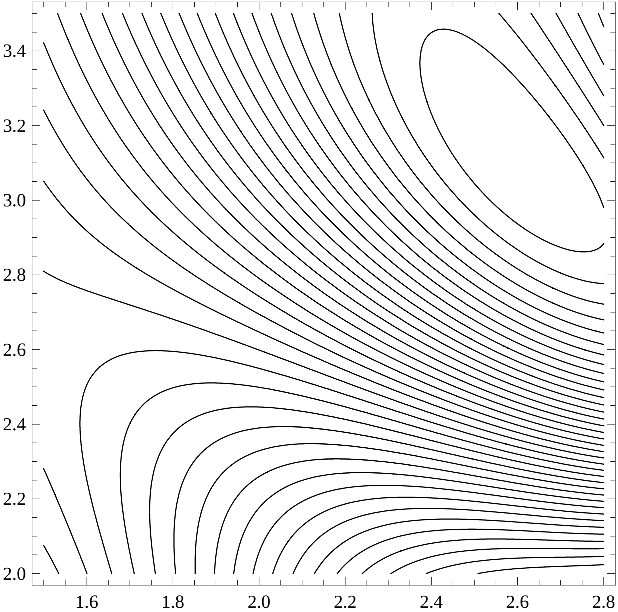
Application: Antropomorphic restoration of curves



Initial family of curves



Restored family of curves



SR problem on the Engel group

- $L = \text{Lie}(X_1, X_2)$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$,
- SR structure on the 4-dim Lie group M:
  $\Delta = \mathrm{span}(X_1, X_2)$, $\qquad \langle X_i, X_j \rangle = \delta_{ij}$, $i, \ j = 1, \ 2$,
- SR problem:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in M, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Results for SR problem on the Engel group

- Parameterization of extremal trajectories,
- Symmetries, Maxwell strata, Maxwell time.

Nilpotent $(2,3,5)$ SR problem

- $L = \text{Lie}(X_1, X_2)$: $[X_1, X_2] = X_3$, $[X_1, X_3] = X_4$, $[X_2, X_3] = X_5$,
- SR structure on the 5-dim Lie group M:
  $\Delta = \mathrm{span}(X_1, X_2)$, $\qquad \langle X_i, X_j \rangle = \delta_{ij}$, $i, \ j = 1, \ 2$,
- SR problem:

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q), \qquad q \in M, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

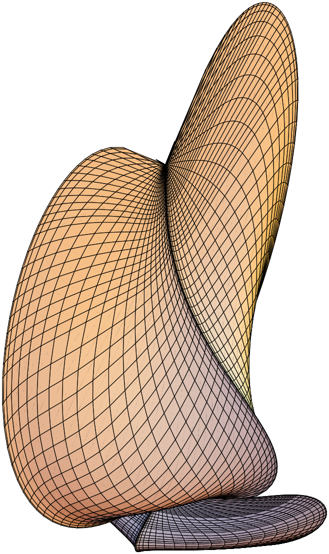
$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$l = \int_0^{t_1} \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Results for nilpotent $(2, 3, 5)$ SR problem

- Parameterization of extremal trajectories,
- Symmetries, Maxwell strata, Maxwell time,
- Bound of conjugate time.

Caustic in nilpotent $(2, 3, 5)$ SR problem



Deciding optimality of extremal trajectories

1. Groups of symmetries $G_{\text{pend}} \supset G_{\text{ad}} \supset G_{\text{Exp}} =: G$
2. Action of G in preimage and image of Exp. Fixed points
3. Maxwell set Max^G. The first Maxwell time t^G_{Max}
4. The bound $t_{\text{conj}} \geq t^G_{\text{Max}}$. Conclusion: $t_{\text{cut}} \leq t^G_{\text{Max}}$.
5. Stratification in the preimage and image of Exp.
6. :) #(doms in preimage) = #(doms in image) $\Rightarrow$ $t_{\text{cut}} = t^G_{\text{Max}}$
   :(#(doms in preimage) > #(doms in image)
   $\Rightarrow$ $t_{\text{cut}} < t^G_{\text{Max}}$ $\Rightarrow$ competing trajectories
7. Reduction of optimal control problem to systems of equations with a unique root in each domain.

Comparing “complexity“ of the optimal control problems

Problem	growth	dim G	extremals	lines	$\mathrm{MAX}^i \cap \mathrm{MAX}^j$	$\frac{\#\mathrm{pre}}{\#\mathrm{im}}$
SE(2)	$(2,3)$	0	Jacobi's	15	0	1
Euler	$(2,3)$	0	Jacobi's	20	1	2
Engel	$(2,3,4)$	1	Jacobi's	25	0	2
Cartan	$(2,3,5)$	2	Jacobi's	35	2	1
S^2 on $\mathbb{R}^2$	$(2,3,5)$	1	Jac, Ell(III)	19	∞	2

Observations and questions

- Why pendulum?
- Any more problems governed by pendulum?
- The case $\#_{\text{pre}} > \#_{\text{im}}$ (Euler, Engel, S^2 on $\mathbb{R}^2$):
  - Non-obvious symmetries and Maxwell strata,
  - Violation of Rolle's theorem for SR problems.
- Countable number of analytic strata in SR sphere (S^2 on $\mathbb{R}^2$) ?
- Deciding optimality: From method to theory?