Image inpainting, neurogeometry of vision, and sub-Riemannian geometry

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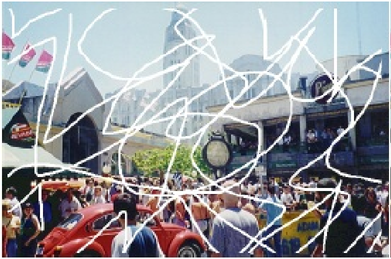
Plan of talk

- Image inpainting
- Model of the primary visual cortex of a human brain (J.Petitot, G.Citti, A.Sarti).
- Problems of sub-Riemannian geometry (A.Agrachev, U.Boscain, F.Rossi) and their solution (Yu.S.)
- Image inpainting via sub-Riemannian length minimizers (Yu.S., A.Ardentov, A.Mashtakov)
- Curve cuspless reconstruction (U.Boscain, R.Duits, F.Rossi, Yu.S.)
- Image inpainting via hypoelliptic diffusion (J.-P.Guthier, U.Boscain, F.Rossi).

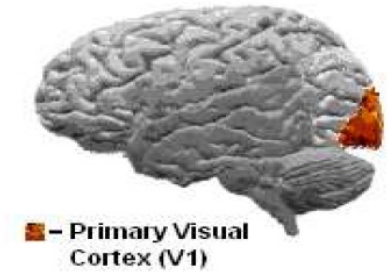
Image inpainting

M. Bertalmio, G. Sapiro, V. Caselles, C. Ballester

Image inpainting



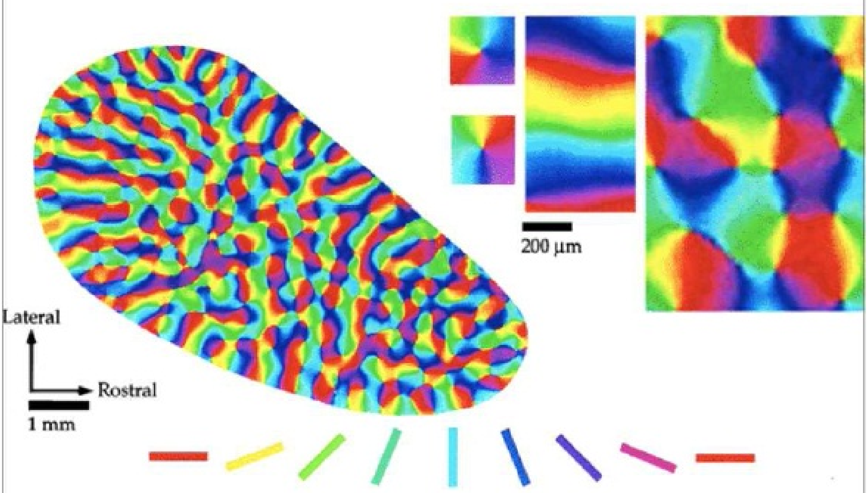
Neurophysiology of vision



A Groups of neurons of the primary visual cortex V1 of human brain are sensible both to position and orientation. Thus V1 lifts images from the plane of image $\mathbb{R}^2$ to the projective tangent bundle $PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1$.

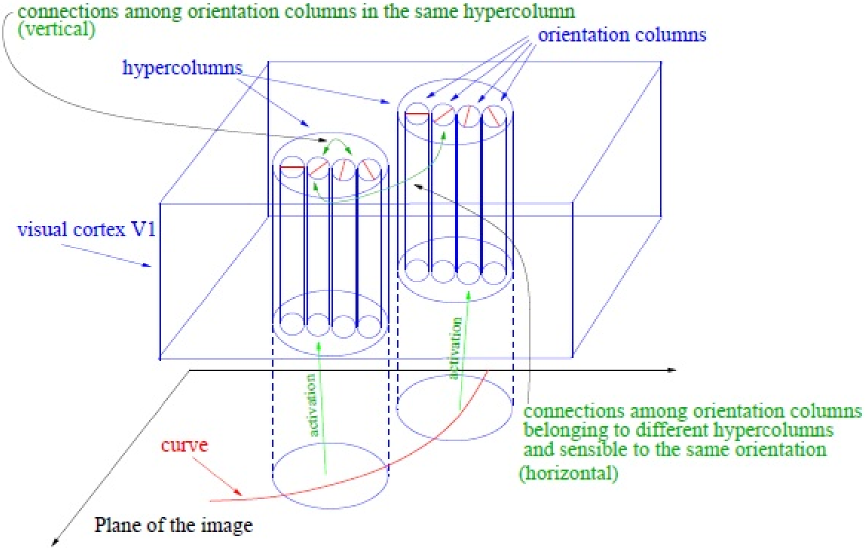
B During inpainting of images, there is minimized the activation energy for neurons not activated by the image at $PT\mathbb{R}^2$.

A1. Hubel and Wiesel (Nobel prise 1981): Groups of neurons sensible to direction



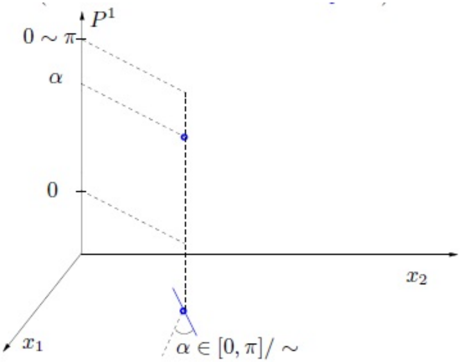
Model of visual cortex V1

“Pinwheel” model:



A2. Lift to $PT\mathbb{R}^2$

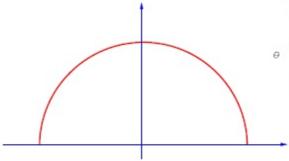
- Brain saves image as a set of positions and directions, i.e., it lifts images to $PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1$.

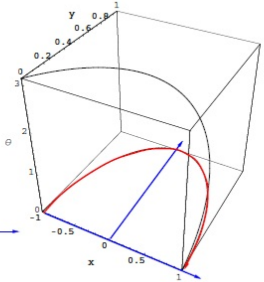


- $PT\mathbb{R}^2$ is a bundle with base $\mathbb{R}^2$ and fiber P^1.

A3. Lift of a curve

- $\mathbb{R}^2 \ni (x(t), y(t)) \mapsto (x(t), y(t), \theta(t)) \in PT\mathbb{R}^2$,
  $\theta(t) = \mathrm{arctg}(\dot{y}(t)/\dot{x}(t)) \in P^1 = [0, \pi]/\sim$.
  Example: $(\cos t, \sin t)$:





- any regular curve in $\mathbb{R}^2$ has a lift to $PT\mathbb{R}^2$,
- not any curve in $PT\mathbb{R}^2$ is a lift of some curve in $\mathbb{R}^2$.

A4. Which curves in $PT\mathbb{R}^2$ are lifts of planar curves?

$$\theta(t) = \mathrm{arctg}(\dot{y}(t)/\dot{x}(t)) \iff$$

$$\dot{x} = u_1 \cos\theta, \quad \dot{y} = u_1 \sin\theta, \quad \dot{\theta} =: u_2,$$

$$q = (x, y, \theta) \in PT\mathbb{R}^2, \qquad u = (u_1, u_2) \in \mathbb{R}^2.$$

$$\dot{q} = u_1 X_1(q) + u_2 X_2(q),$$

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

B1. Which functional is minimized?

- brain minimizes a functional (internal or external for the brain),
- when moving a hand, the brain minimizes a compromise between energy and strength of muscles (exterior functional),
- when reconstructing a curve, the brain minimizes the activation energy of neurons (interior functional),
- easily activated are the neurons close one to another both in position and in orientation (i.e., close in $PT\mathbb{R}^2$).

Problem of sub-Riemannian geometry on $PT\mathbb{R}^2$

$$\int (u_1^2 + \alpha^2 u_2^2)\, dt \to \min \iff \int \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min$$

$$\dot q = u_1 X_1(q) + u_2 X_2(q), \quad q = (x, y, \theta) \in PT\mathbb{R}^2, \quad u = (u_1, u_2) \in \mathbb{R}^2,$$
$$q(0) = q_0, \qquad q(t_1) = q_1,$$
$$\int_0^{t_1} \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min .$$

$$\theta \in P^1 = \mathbb{R}/(\pi\mathbb{Z}) = [0, \pi]/\sim .$$

Problem of sub-Riemannian geometry on SE(2)

$$\mathrm{SE}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta & x \\ \sin\theta & \cos\theta & y \\ 0 & 0 & 1 \end{pmatrix} \mid \theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z}), \ x, y \in \mathbb{R} \right\} \cong$$

$$\cong \mathbb{R}^2 \times S^1.$$

$$\theta \in S^1 = \mathbb{R}/(2\pi\mathbb{Z}) = [0, 2\pi]/\sim .$$

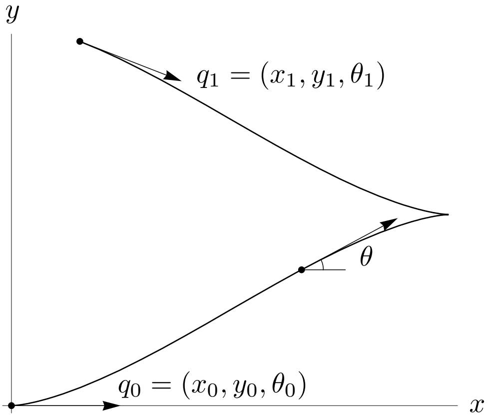
$$\dot q = u_1 X_1(q) + u_2 X_2(q), \quad q = (x, y, \theta) \in \mathrm{SE}(2), \quad u = (u_1, u_2) \in \mathbb{R}^2,$$

$$q(0) = q_0, \qquad q(t_1) = q_1,$$

$$X_1(q) = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \qquad X_2(q) = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\int_0^{t_1} \sqrt{u_1^2 + \alpha^2 u_2^2}\, dt \to \min .$$

Sub-Riemannian problem
on the group of motions of a plane, *or*
Problem on optimal motion of a mobile robot in the plane

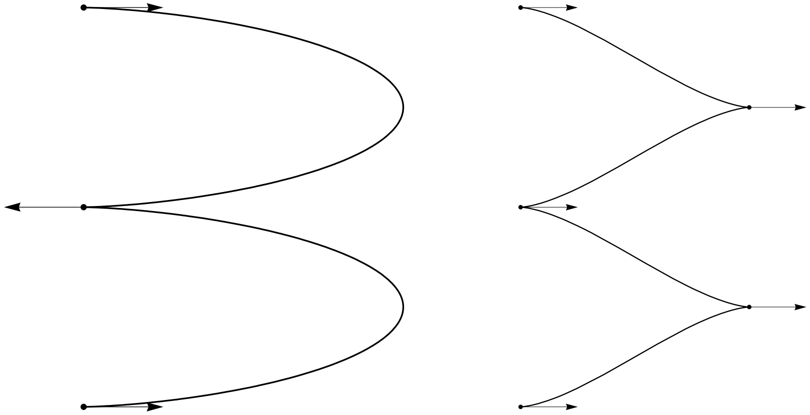


$$q(0) = q_0, \qquad q(t_1) = q_1, \qquad l = \int_0^{t_1} \sqrt{\dot{x}^2 + \dot{y}^2 + \alpha^2\,\dot{\theta}^2}\,dt \to \min$$

Results on sub-Riemannian problems on SE(2) and $PT\mathbb{R}^2$

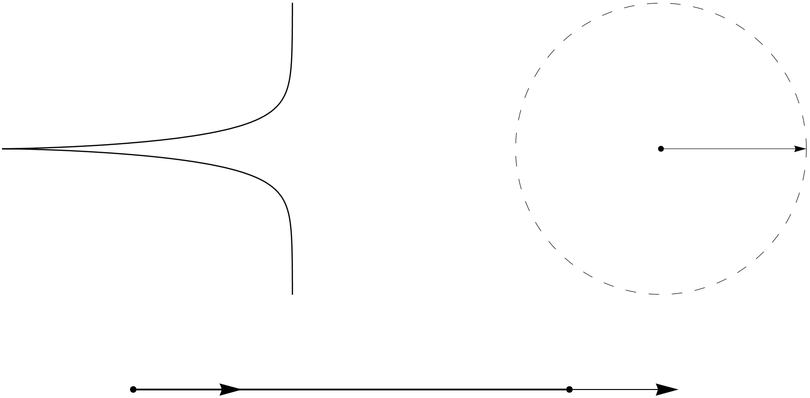
- Existence of optimal trajectories,
- Parameterisation of extremal trajectories (via Pontryagin maximum principle),
- Description of optimal trajectories:
    - Arbitrary boundary conditions $\Rightarrow$ reduction to systems of algebraic equations,
    - Special boundary conditions $\Rightarrow$ explicit solutions,
- Structure of optimal synthesis and Maxwell set,
- Sub-Riemannian spheres,
- Applications: reconstruction of curves, Parallel software for image inpainting.

Sub-Riemannian problem on SE(2): generic extremal trajectories



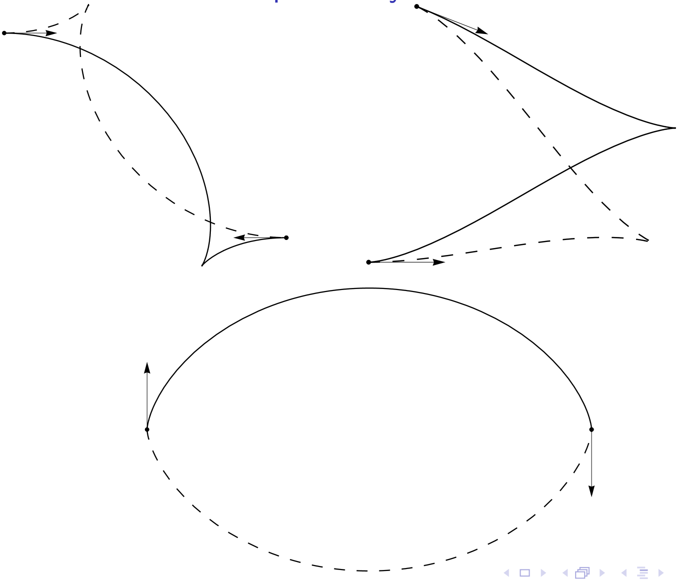
Parameterisation by Jacobi's functions cn, sn, dn, E.

Sub-Riemannian problem on SE(2): special extremal trajectories



Parameterisation by elementary functions.

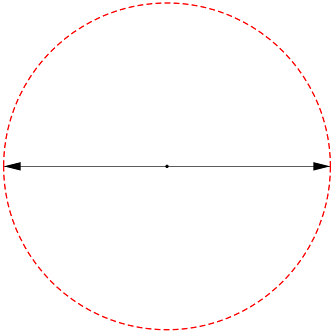
Maxwell points on extremal trajectories: optimality loss



Optimal trajectories

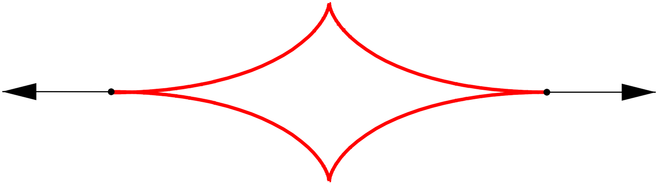
$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = 0$$

$$x_1 = 0, \qquad y_1 = 0, \qquad \theta_1 \neq 0$$



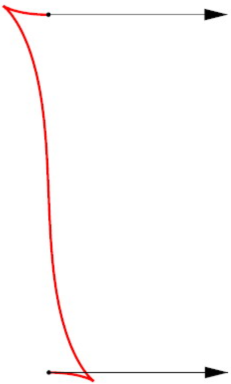
Optimal trajectories

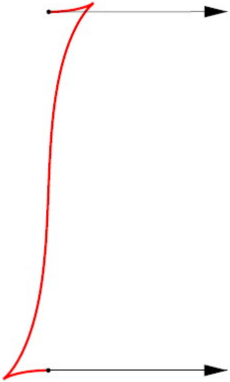
$$x_1 \neq 0, \qquad y_1 = 0, \qquad \theta_1 = \pi$$



Optimal trajectories

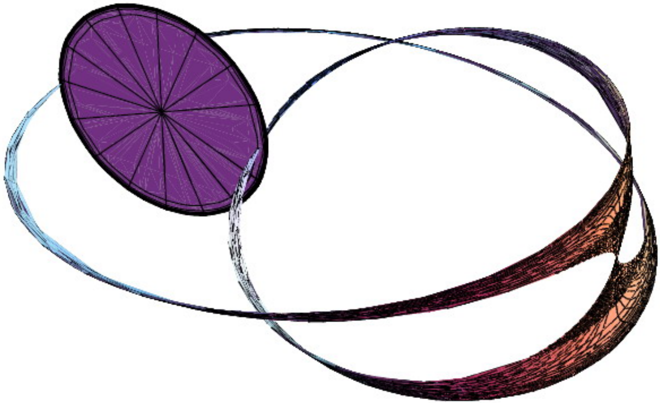
$$x_1 = 0, \qquad y_1 \neq 0, \qquad \theta_1 = 0$$





Maxwell set

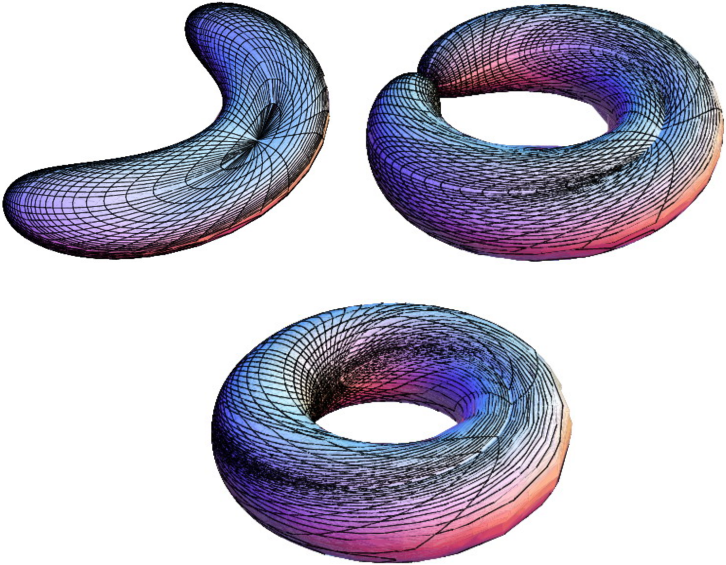
$$\mathrm{Max} = \{q_1 \in G \mid \exists > 1 \text{ optimal trajectories } q(\cdot) \ : \ q(t_1) = q_1\}$$



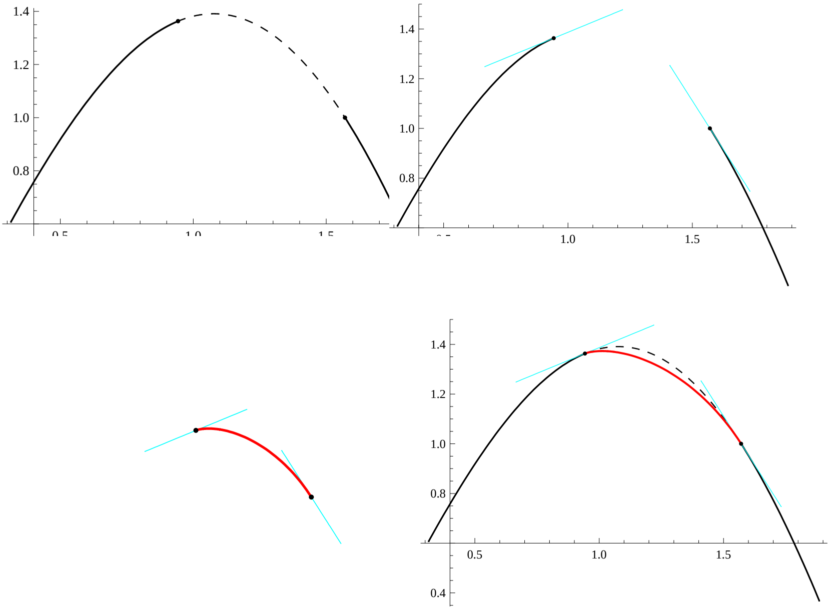
Sub-Riemannian metric and spheres

- $d(q_0, q_1) = \inf\{l(q(\cdot)) \mid q(0) = q_0,\ q(t_1) = q_1\}$
- $S_R = \{q \in G \mid d(q_0, q) = R\}$
- $R = 0 \quad \Rightarrow \quad S_R = \{q_0\}$
- $R \in (0, \pi) \quad \Rightarrow \quad S_R \cong S^2$
- $R = \pi \quad \Rightarrow \quad S_R \cong S^2/\{N = S\}$
- $R > \pi \quad \Rightarrow \quad S_R \cong \mathbb{T}^2$

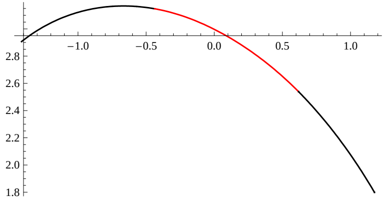
Global structure of sub-Riemannian spheres in SE(2)

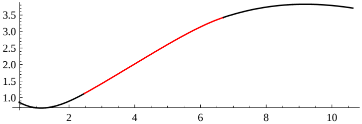


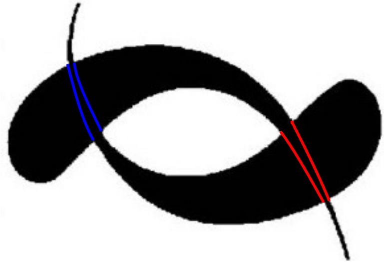
Reconstruction of corrupted curve

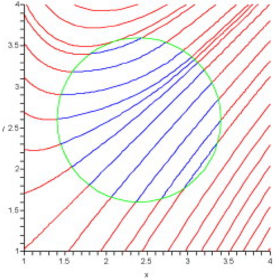


Reconstruction of corrupted curve

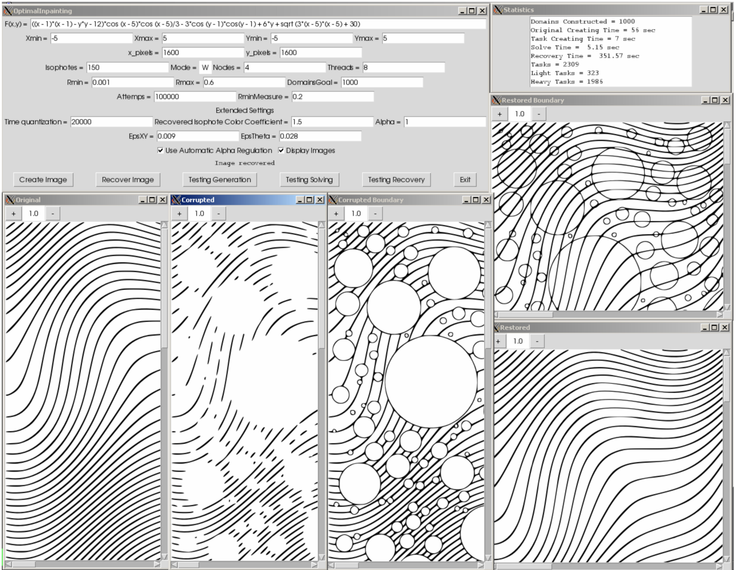




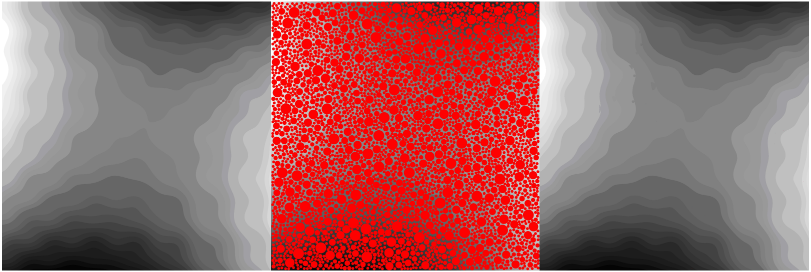




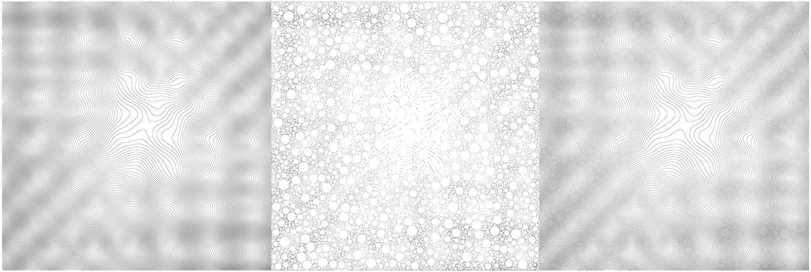
Parallel software `OptimalInpainting` for corrupted images reconstruction



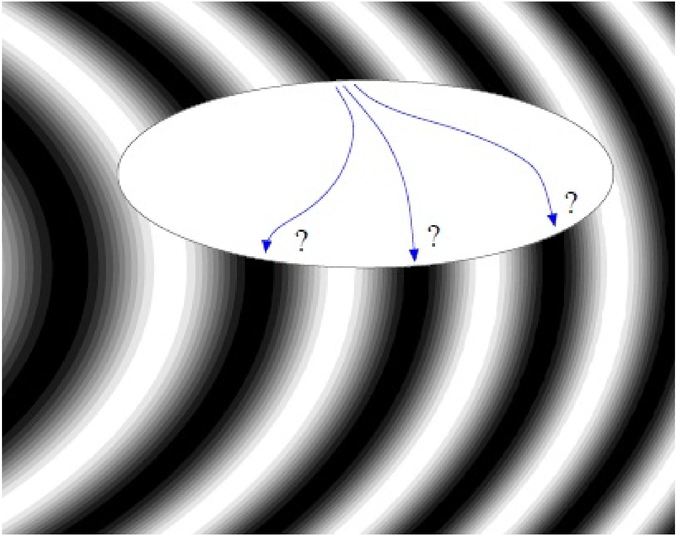
Original, corrupted, and reconstructed grayscale image



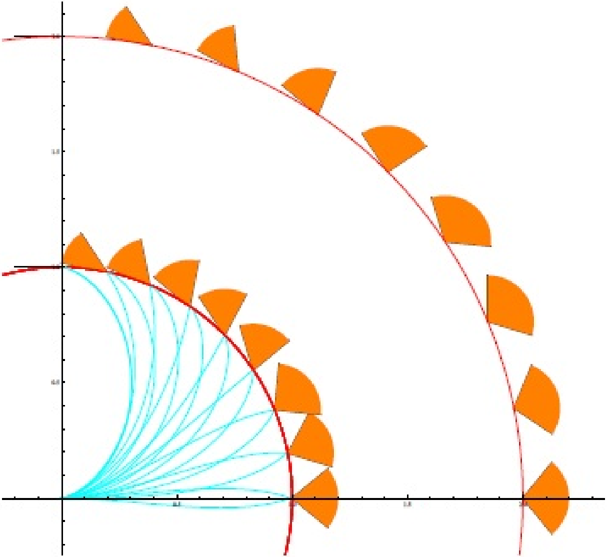
Original, corrupted, and reconstructed binary image



Problem 1: non-uniqueness of level sets



Problem 2: cusp points



Problem 3: extraction of isophotes (level lines of brightness) from half-tone image

Problem 4: critical points of brightness

Image inpainting via hypoellpitic diffusion on $PT\mathbb{R}^2$

Corrupted image $I : D \setminus \Omega \to [0, +\infty)$

1. Smoothing $f = I * G_\sigma$:
   $f(x, y) = \iint_{\mathbb{R}^2} I(\tilde{x}, \tilde{y}) G_\sigma(x - \tilde{x}, y - \tilde{y})\, d\tilde{x}\, d\tilde{y}$,
   $G_\sigma(x, y) = \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$.
2. Lift $f : \mathbb{R}^2 \to \mathbb{R}$ to $\overline{f} : PT\mathbb{R}^2 \to \mathbb{R}$:
   $$\overline{f}(x, y, \theta) = \begin{cases} f(x, y), \text{ if } \theta \text{ is the slope of } \{f = \text{const}\}, \\ 0 \text{ else} \end{cases}$$
3. Hypoelliptic (anisotropic) diffusion
   $$\partial_t \Phi(q, t) = (X_1^2 + X_2^2)\Phi(q, t),$$
   $$\Phi(q, 0) = \overline{f}(q)$$
4. Projection to $\mathbb{R}^2$:
   $$\tilde{f}(x, y) = \max_{\theta \in P^1} \Phi(x, y, \theta, T).$$

Isotropic diffusion in $\mathbb{R}^3$

- Heat equation

$$\partial_t \Phi(x,y,z,t) = (\partial_x^2 + \partial_y^2 + \partial_z^2)\Phi(x,y,z,t),$$
$$\Phi(x,y,z,0) = \varphi(x,y,z).$$

- Fundamental solution (heat kernel)

$$\partial_t \mathcal{E}(x,y,z,t) - (\partial_x^2 + \partial_y^2 + \partial_z^2)\mathcal{E}(x,y,z,t) = \delta(x,y,z,t),$$
$$\mathcal{E}(x,y,z,t) = \frac{\theta(t)}{(2\sqrt{\pi t})^3} \exp\left(-\frac{x^2+y^2+z^2}{4t}\right),$$
$$\Phi = \varphi * \mathcal{E}.$$

- Propagation of isotropic diffusion along Riemannian geodeics

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix} = u_1\partial_x + u_2\partial_y + u_3\partial_z,$$
$$\int \sqrt{u_1^2 + u_2^2 + u_3^2}\, dt \to \min .$$

Anisotropic diffusion in $PT\mathbb{R}^2$

- PDE of anisotropic diffusion

$$\partial_t \Phi(q,t) = (X_1^2 + X_2^2)\Phi(q,t),$$
$$\Phi(q,0) = \varphi(q), \qquad q = (x,y,\theta) \in PT\mathbb{R}^2 = \mathbb{R}^2 \times P^1.$$

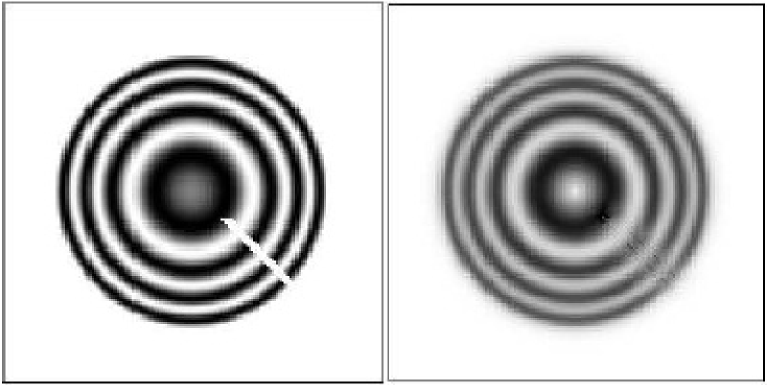
- Fundamental solution (kernel of anisotropic diffusion)

$$\partial_t \mathcal{E}(q,t) - (X_1^2 + X_2^2)\mathcal{E}(q,t) = \delta(q,t),$$
$$\mathcal{E}(q,t) = \dots,$$
$$\Phi = \varphi * \mathcal{E}.$$

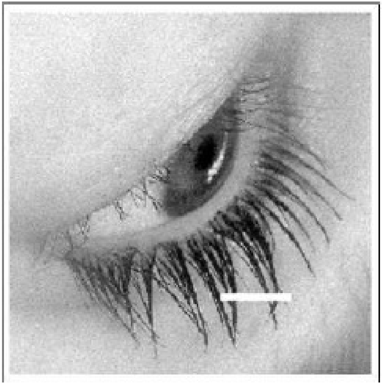
- Propagation of anisotropic diffusion along sub-Riemannian geodeics

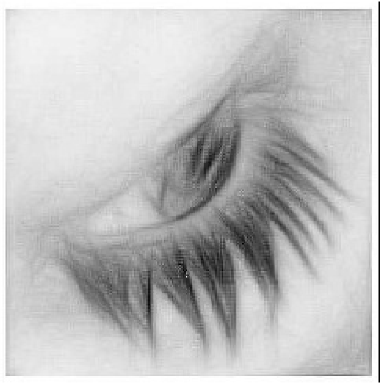
$$\dot{q} = u_1 X_1(q) + u_2 X_2(q),$$
$$\int \sqrt{u_1^2 + u_2^2}\, dt \to \min .$$

Anisotropic diffusion: experiments



Anisotropic diffusion: experiments





Anisotropic diffusion: experiments